On-line Appendix Material
Monetary Policy Trade-offs at the Zero Lower Bound*

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Abstract

This appendix contains additional detail, derivations, and robustness exercises.

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1 The New Keynesian Model

We use a simple New Keynesian model to study zero interest rate policy. This facilitates analytical results and comparison to other recent papers on this topic. Some assumptions are made for expositional simplicity — for example linear disutility of labor supply. Further details on the microfoundations can be found in Woodford (2003) and Gali (2008).

1.1 Optimal Decisions

A continuum of households i on the unit interval maximize utility

$$\hat{E}_t^i \sum_{T=t}^{\infty} \bar{C}_T \beta^{T-t} \left[\frac{(c_T(i))^{1-1/\sigma}}{(1-1/\sigma)} - \chi n_T(i) \right],$$

where $0 < \beta < 1$, $\sigma > 0$ and $\chi > 0$, by choice of sequences for consumption, $c_t(i)$, and labor supply, $n_t(i)$, subject to the flow budget constraint

$$c_t(i) + b_t(i) \leq (1 + R_{t-1}) \pi_t^{-1} b_{t-1}(i) + W_t n_t(i)/P_t + \Gamma_t(i)/P_t$$

and the No-Ponzi condition

$$\lim_{T\to\infty} \hat{E}_t^i \left(\prod_{s=0}^{T-t} (1 + R_{t+s}) \pi_{t+s+1}^{-1} \right)^{-1} b_{T+1}(i) \geq 0.$$

The variable $b_t(i) \equiv B_t(i)/P_t$ denotes real bond holdings (which in equilibrium are in zero net supply), R_t the nominal interest rate, $\pi_t \equiv P_t/P_{t-1}$ the inflation rate, W_t is the hourly wage, $\Gamma_t(i)$ dividends from equity holdings of firms and $\bar{C}_T$ exogenous preference shifter. The operator $\hat{E}_t^i$ denotes subjective expectations, which might differ from rational expectations.

A continuum of monopolistically competitive firms maximize profits

$$\hat{E}_t^j \sum_{T=t}^{\infty} \xi^{T-t} Q_{t,T} \left[p_t(j) y_T(j) - W_T n_T(j) \right]$$

by choice of $p_t(j)$ subject to the production technology and demand function $y_T(j) = n_T(j) = (p_t(j)/P_T)^{-\theta} y_T$ for all $T \geq t$, with the elasticity of demand across differentiated goods an exogenous process satisfying $\theta > 1$; and exogenous probability $0 < \xi < 1$ of not being able to reset their price in any subsequent period. When setting prices in period t, firms are assumed to value future streams of income at the marginal value of aggregate income in terms of the marginal value of an additional unit of aggregate income today giving the stochastic discount factor $Q_{t,T} = \beta^{T-t}\left[(P_t y_t)/(P_T y_T)\right]^{(1/\sigma)}$.

For any beliefs satisfying standard probability laws, to a first-order log-linear approximation in the neighborhood of a zero-inflation steady state, optimal individual consumption and pricing decisions can be expressed as

$$\hat{c}_t(i) = \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t}\left[(1-\beta)\hat{w}_T - \beta\sigma\left(\hat{R}_T - \hat{\pi}_{T+1} - (\bar{c}_T - \bar{c}_{T+1})\right)\right] \quad (1)$$

$$\hat{p}_t(j) = \hat{E}_t^j \sum_{T=t}^{\infty} (\xi\beta)^{T-t}\left[(1-\xi\beta)\hat{w}_T + \xi\beta\hat{\pi}_{T+1}\right] \quad (2)$$

where for any variable z_t, $\hat{z}_t = \ln(z_t/\bar{z})$ the log-deviation from steady state $\bar{z}$, with the exceptions $\hat{p}_t(j) = \ln(p_t(j)/P_t)$, $\hat{R}_t = \ln\left[(1+R_t)/\left(1+\bar{R}\right)\right]$, and $\bar{c}_t = \ln\left(\bar{C}_t/\bar{C}\right)$. With a slight abuse of notation, the caret denoting log deviation from steady state is dropped for the remainder, so long as no confusion results.

In a symmetric equilibrium $c_t(i) = c_t = w_t \equiv W_t/P_t = n_t = y_t$ for all i, $p_t(j) = p_t(j)$ and $b_t(i) = b_t(j) = 0$ for all i, j.[1] Aggregating across the continuum of households and firms, and imposing market-clearing conditions, the economy is described by the aggregate demand and supply equations

$$x_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\left[(1-\beta)x_{T+1} - \sigma\left(R_T - \pi_{T+1} - r_T^n\right)\right] \quad (3)$$

$$\pi_t = \hat{E}_t \sum_{T=t}^{\infty} (\xi\beta)^{T-t}\left[\kappa x_T + (1-\xi)\beta\pi_{T+1}\right] \quad (4)$$

where the output gap is defined as

$$x_t = y_t - y_t^n = w_t$$

the difference between output and the natural rate of output, the level of output determined by a flexible price economy: here $y_t^n = 0$. We assume agents understand this equilibrium relationship between wages and the output gap. This is without loss of generality. The associated natural rate of interest $r_t^n = (\bar{c}_t - \hat{E}_t\bar{c}_{t+1})$ is determined by fluctuations in the propensity to consume, an exogenous process to be discussed. Average beliefs are defined as

$$\int_0^1 \hat{E}_t^i di = \int_0^1 \hat{E}_t^j dj = \hat{E}_t.$$

[1] The optimal consumption decision rule implictly made use of this equilibrium requirement.

The aggregate demand equation determines the output gap as the discounted expected value of future wages, with the second term capturing variations in the real interest rate, applied in future periods, due to changes in the nominal interest rate and goods price inflation. That expected future dividends are irrelevant to consumption plans, to the first-order, reflects the assumption of an infinite Frisch elasticity of labor supply.[2] The aggregate supply curve determines inflation as the discounted future sequence of marginal costs and the inflation rate. The slope of the Phillips curve is measured by $\kappa = (1-\xi\beta)(1-\xi)/\xi$.

2 Forecasting model

Agents form estimates of equilibrium macroeconomic variables like econometricians. Expectations are based on the forecasting model

$$z_t = z_S + \bar{\omega}_t + e_t \tag{5}$$

$$z_S = \Omega_S \bar{r} \tag{6}$$

$$\bar{\omega}_{t+1} = \rho\bar{\omega}_t + u_{t+1} \tag{7}$$

where $S \in [H, L]$ and

$$z_t = \begin{bmatrix} \pi_t \\ x_t \end{bmatrix}, \quad z_S = \begin{bmatrix} \pi_S \\ x_S \end{bmatrix}, \text{ and } \bar{\omega}_t = \begin{bmatrix} \bar{\omega}_t^\pi \\ \bar{\omega}_t^x \end{bmatrix}$$

and $0 \leq \rho \leq 1$ a parameter; e_t and u_t i.i.d. with $J = E\left[e_t e_t'\right]$, $Q = E\left[u_t u_t'\right]$ and Ω_S prior beliefs about the consequences of a negative demand shock. These priors are invariant to the state.

Beliefs are updated using the the Kalman filter. Consider the recursion

$$\begin{aligned} \omega_{t+1|t} &= \rho\omega_{t|t-1} + \rho P_t \left(P_t + J\right)^{-1} \mathcal{F}_t \\ P_{t+1} &= \rho^2 P_t - \rho^2 P_t \left(P_t + J\right)^{-1} P_t + Q \end{aligned}$$

where the matrix P_t is the mean square error associated with the estimate ω_{t+1}. The vector $\mathcal{F}_t$ denotes the current prediction error

$$\mathcal{F}_t = \left(z_t - z_S - \omega_{t|t-1}\right).$$

Following Sargent and Williams (2005), we make the following simplifying assumptions.

[2]This assumption doesn't have any further implications. Our quantitative analysis calibrates the slope of aggregate supply curve directly.

Re-scale the posterior estimate as $P_t = \Xi_t J$ and use the approximation $(I+\Xi_t)^{-1} \simeq I$ for small Ξ_t to give

$$\begin{aligned}\omega_{t+1|t} &= \rho\omega_{t|t-1} + \rho\Xi_t\mathcal{F}_t \\ \Xi_{t+1} &= \rho^2\Xi - \rho^2\Xi_t\Xi_t + QJ^{-1}.\end{aligned}$$

We restrict the analysis to the steady state of this filter assuming prior beliefs satisfy the restriction $Q = \hat{c}^2 J$ for scalar $\hat{c}$. We want to solve for Ξ in

$$\Xi\Xi + (1-\rho^2)\Xi - \hat{c}^2 I = 0.$$

The solution has the form $\Xi = bI$ where α solves

$$\gamma^2 I + (1-\rho^2)\gamma - \hat{c}^2 I = 0.$$

to give

$$\gamma = \frac{-(1-\rho^2) + \sqrt{(1-\rho^2)^2 + 4\hat{c}^2}}{2}.$$

Under these assumptions the updating equation becomes

$$\omega_{t+1|t} = \rho\omega_{t|t-1} + \rho\gamma\mathcal{F}_t$$

where $0 < \gamma < 1$ is a function of the parameters ρ and $\hat{c}$. The leaning gain is then $g \equiv \rho\gamma$. In the special (and more common) case $\rho = 1$ and $g \equiv \hat{c}$.

3 Proof of Proposition 1

Wtih $\rho = 1$ the system has characteristic polynomial:

$$\begin{aligned}P(X) &= X^2 + \left(-\beta\frac{\alpha-1}{\alpha\beta-1} - \kappa\frac{\sigma}{\beta(\delta-1)+1} - 1\right)X \\ &\quad + \left(\beta\frac{\alpha-1}{\alpha\beta-1} + \kappa\frac{\sigma}{\beta(\delta-1)+1} + \kappa\sigma\frac{\gamma^2}{(\beta(\delta-1)+1)(\alpha\beta-1)}\right).\end{aligned}$$

Because

$$P(1) = \kappa\sigma\frac{\gamma^2}{(\alpha\beta-1)(\beta\delta-\beta+1)} < 0$$

there must be at least one eigenvalue outside the unit circle. By continuity there must be some ρ^* such that for $\rho^* \leq \rho \leq 1$ there continues to be at least one eigenvalue outside the unit circle.

4 Some basics: Conditional expectations

To derive the optimal decision rules, some basic properties of expectations are helpful. We start assuming that the economy reverts to the high state with certainty in period $\bar{T}$. This assumption is central to our approximation of the true model in which $\bar{T} \to \infty$. We therefore compute the true model by solving for optimal decisions conditional for an arbitrary $\bar{T}$ then take limits. For finite $\bar{T}$ the rational expectations solution of the model is time varying and denoted by $z_{S,t}^{RE}$. The following derivations assume that the constant in agents' beliefs is always equal to this rational expectations solution. For ease of notation, we do not make explicit the dependence on $\bar{T}$.

Conditional on being in the high state output expectations satisfy

$$E_t x_{T+1} \quad = \quad x_{H,t}^{RE} + \rho^{T-t} \omega_{t|t-1}^{x}$$

so that

$$E_t \sum_{T=t}^{\infty} \beta^{T-t} \left(x_{H,t}^{RE} + \rho^{T-t} \omega_{t|t-1}^{x} \right) = \frac{1}{1-\beta} x_{H,t}^{RE} + \frac{1}{1-\beta\rho} \omega_{t|t-1}^{x}.$$

Similar expressions hold for inflation. For the nominal interest rate we have

$$E_t R_{T+1} = r_H + \rho^{T-t} \omega_{t|t-1}^{\pi}$$

and therefore

$$E_t \sum_{T=t}^{\infty} \beta^{T-t} \left(r_H + \rho^{T-t} \omega_{t|t-1}^{\pi} \right) = \frac{1}{1-\beta} r_H + \frac{1}{1-\beta\rho} \omega_{t|t-1}^{\pi}.$$

Conditional on being in the low state, expectations satisfy the following relationships.

$$E_t x_{T+1} = (1-\delta)^{T+1-t} x_{L,t}^{RE} + \left(1 - (1-\delta)^{T+1-t} \right) x_{H,t}^{RE} + \rho^{T-t} \omega_{t|t-1}^{x}.$$

Suppose after $\bar{T}$ the economy returns to the high state with probability one. Then

$$
\begin{aligned}
E_t \sum_{T=t}^{\bar{T}} \beta^{T-t} x_{T+1} &= \sum_{T=t}^{\bar{T}} \beta^{T-t} (1-\delta)^{T+1-t} x_t^L + \sum_{T=t}^{\bar{T}} \beta^{T-t} \left(1-(1-\delta)^{T+1-t}\right) x_{H,t}^{RE} + \sum_{T=t}^{\bar{T}} \beta^{T-t} \rho^{T-t} \omega_{t|t-1}^x \\
&= (1-\delta) \frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)} x_{L,t}^{RE} + \left[\frac{1-\beta^{\bar{T}+1-t}}{1-\beta} - (1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}\right] x_{H,t}^{RE} \\
&\quad + \frac{1-(\beta\rho)^{\bar{T}+1-t}}{1-\beta\rho} \omega_{t|t-1}^x .
\end{aligned}
$$

If x_t a constant with $\omega_{t|t-1} = 0$ then

$$
\begin{aligned}
E_t \sum_{T=t}^{\bar{T}} \beta^{T-t} x_{T+1} &= (1-\delta) \frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)} x + \left[\frac{1-\beta^{\bar{T}+1-t}}{1-\beta} - (1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}\right] x \\
&= \left[(1-\delta) \frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)} - (1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}\right] x + \left[\frac{1-\beta^{\bar{T}+1-t}}{1-\beta}\right] x \\
&= \left[\frac{1-\beta^{\bar{T}+1-t}}{1-\beta}\right] x .
\end{aligned}
$$

Now consider the infinite horizon case where we are in the high state with probability 1 after $t > \bar{T}$. Then

$$
\begin{aligned}
E_t \sum_{T=t}^{\infty} \beta^{T-t} x_{T+1} &= E_t \sum_{T=t}^{\bar{T}} \beta^{T-t} x_{T+1} + E_t \sum_{T=\bar{T}+1}^{\infty} \beta^{T-t} x_{T+1} \\
&= (1-\delta) \frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)} x_{L,t}^{RE} + \left[\frac{1-\beta^{\bar{T}+1-t}}{1-\beta} - (1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}\right] x_{H,t}^{RE} \\
&\quad + \frac{1-(\beta\rho)^{\bar{T}+1-t}}{1-\beta\rho} \omega_{t|t-1}^x + E_t \sum_{T=\bar{T}+1}^{\infty} \beta^{T-t} \left(x_{H,t}^{RE} + \rho^{T-t} \omega_{t|t-1}^x\right) \\
&= (1-\delta) \frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)} x_{L,t}^{RE} + \left[\frac{1-\beta^{\bar{T}+1-t}}{1-\beta} - (1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}\right] x_{H,t}^{RE} \\
&\quad + \frac{1-(\beta\rho)^{\bar{T}+1-t}}{1-\beta\rho} \omega_{t|t-1}^x + \frac{\beta^{\bar{T}+1-t}}{1-\beta} x_t^H + \frac{(\beta\rho)^{\bar{T}+1-t}}{1-\beta\rho} \omega_{t|t-1}^x \\
&= (1-\delta) \frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)} x_{L,t}^{RE} + \left[\frac{1}{1-\beta} - (1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}\right] x_{H,t}^{RE} \\
&\quad + \frac{1}{1-\beta\rho} \omega_{t|t-1}^x .
\end{aligned}
$$

Similar expressions hold for inflation.

For the interest rate we have

$$E_t R_{T+1} = \left(1 - (1-\delta)^{T+1-t}\right)\left(r_H + \rho^{T-t}\omega^{\pi}_{t|t-1}\right)$$

and

$$\begin{aligned} E_t \sum_{T=t}^{\bar{T}} \beta^{T-t} R_{T+1} &= \sum_{T=t}^{\bar{T}} \beta^{T-t}\left(1 - (1-\delta)^{T+1-t}\right)\left(r_H + \rho^{T-t}\omega^{\pi}_{t|t-1}\right) \\ &= \left[\frac{1-\beta^{\bar{T}+1-t}}{1-\beta} - (1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}\right] r_H \\ &\quad + \left[\frac{1-(\beta\rho)^{\bar{T}+1-t}}{1-\beta\rho} - (1-\delta)\frac{1-[\beta\rho(1-\delta)]^{\bar{T}+1-t}}{1-\beta\rho(1-\delta)}\right]\omega^{\pi}_{t|t-1}. \end{aligned}$$

The infinite sum is then

$$\begin{aligned} E_t \sum_{T=t}^{\infty} \beta^{T-t} R_{T+1} &= E_t \sum_{T=t}^{\bar{T}} \beta^{T-t} R_{T+1} + E_t \sum_{T=\bar{T}+1}^{\infty} \beta^{T-t} R_{T+1} \\ &= \left[\frac{1-\beta^{\bar{T}+1-t}}{1-\beta} - (1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}\right] r_H \\ &\quad + \left[\frac{1-(\beta\rho)^{\bar{T}+1-t}}{1-\beta\rho} - (1-\delta)\frac{1-[\beta\rho(1-\delta)]^{\bar{T}+1-t}}{1-\beta\rho(1-\delta)}\right]\omega^{\pi}_{t|t-1} \\ &\quad + \frac{\beta^{\bar{T}+1-t}}{1-\beta} r_H + \frac{(\beta\rho)^{\bar{T}+1-t}}{1-\beta\rho}\omega^{\pi}_{t|t-1} \\ &= \left[\frac{1}{1-\beta} - (1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}\right] r_H \\ &\quad + \left[\frac{1}{1-\beta\rho} - (1-\delta)\frac{1-[\beta\rho(1-\delta)]^{\bar{T}+1-t}}{1-\beta\rho(1-\delta)}\right]\omega^{\pi}_{t|t-1} \end{aligned}$$

5 Dynamics without forward Guidance

In all the models we consider the rational expectations solution under optimal discretion satisfies $\pi^{RE}_{H,t} = x^{RE}_{H,t} = 0$. We impose this restriction in some expressions that follow. However, in some cases it is left to show the derivations in greater generality.

5.1 High state solution

In the high state we have

$$\begin{aligned} x_t &= E_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} - \sigma \left(R_T - \pi_{T+1} - r_T \right) \right] \\ &= -\sigma \left(R_t - r_H \right) + \frac{1-\beta}{1-\beta\rho} \omega^x_{t|t-1} + \sigma \frac{1-\beta}{1-\beta\rho} \omega^\pi_{t|t-1} \end{aligned}$$

and

$$\begin{aligned} \pi_t &= E_t \sum_{T=t}^{\infty} (\xi\beta)^{T-t} \left[\kappa x_T + (1-\xi)\beta \pi_{T+1} \right] \\ &= \kappa x_t + \frac{\kappa\xi\beta}{1-\xi\beta\rho} \omega^x_{t|t-1} + \frac{(1-\xi)\beta}{1-\xi\beta\rho} \omega^\pi_{t|t-1}. \end{aligned}$$

These expressions make use of the properties of rational expectations equilibrium that in the high state $x^{RE}_{H,t} = \pi^{RE}_{H,t} = 0$.

5.2 Low state solution: Arbitrary $\bar{T}$

Consider the solution when the natural rate reverts with probability one after $\bar{T}$:

$$\begin{aligned} x_t &= E_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} - \sigma \left(R_T - \pi_{T+1} - r_T \right) \right] \\ &= -\sigma \left(R_t - r_t \right) + E_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} - \sigma \left(\beta R_{T+1} - \pi_{T+1} - \beta r_{T+1} \right) \right] \\ &= \sigma r_L + \\ & (1-\beta) \left[(1-\delta) \frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)} x^{RE}_{L,t} \right] + \\ & + (1-\beta) \left[\left[\frac{1}{1-\beta} - (1-\delta) \frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)} \right] x^{RE}_{H,t} + \frac{1}{1-\beta\rho} \omega^x_{t|t-1} \right] + \\ & -\sigma\beta \left[\left[\frac{1}{1-\beta} - (1-\delta) \frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)} \right] r_H + \left[\frac{1}{1-\beta\rho} - (1-\delta) \frac{1-[\beta\rho(1-\delta)]^{\bar{T}+1-t}}{1-\beta\rho(1-\delta)} \right] \omega^\pi_{t|t-1} \right] \\ & +\sigma \left[(1-\delta) \frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)} \pi^{RE}_{L,t} + \left[\frac{1}{1-\beta} - (1-\delta) \frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)} \right] \pi^{RE}_{H,t} + \frac{1}{1-\beta\rho} \omega^\pi_{t|t-1} \right] \\ & +\sigma\beta \left[(1-\delta) \frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)} r_L + \left[\frac{1}{1-\beta} - (1-\delta) \frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)} \right] r_H \right] \end{aligned}$$

where the third equality employs various properties of expectations discussed earlier.

Now aggregate supply

$$\begin{aligned}
\pi_t &= E_t \sum_{T=t}^{\infty} (\xi\beta)^{T-t} \left[\kappa x_T + (1-\xi)\,\beta\pi_{T+1}\right] \\
&= \kappa x_t + E_t \sum_{T=t}^{\infty} (\xi\beta)^{T-t} \left[\kappa\xi\beta x_{T+1} + (1-\xi)\,\beta\pi_{T+1}\right] \\
&= k x_t + \kappa\xi\beta \left[\frac{1}{1-\xi\beta\rho}\omega^x_{t|t-1}\right] + (1-\xi)\beta\left[\frac{1}{1-\xi\beta\rho}\omega^\pi_{t|t-1}\right] + \\
&\quad \kappa\xi\beta\left[(1-\delta)\frac{1-\left[\xi\beta(1-\delta)\right]^{\bar{T}+1-t}}{1-\xi\beta(1-\delta)}x^{RE}_{L,t} + \left[\frac{1}{1-\xi\beta} - (1-\delta)\frac{1-\left[\xi\beta(1-\delta)\right]^{\bar{T}+1-t}}{1-\xi\beta(1-\delta)}\right]x^{RE}_{H,t}\right] \\
&\quad + (1-\xi)\,\beta\left[(1-\delta)\frac{1-\left[\xi\beta(1-\delta)\right]^{\bar{T}+1-t}}{1-\xi\beta(1-\delta)}\pi^{RE}_{L,t} + \left[\frac{1}{1-\xi\beta} - (1-\delta)\frac{1-\left[\xi\beta(1-\delta)\right]^{\bar{T}+1-t}}{1-\xi\beta(1-\delta)}\right]\pi^{RE}_{H,t}\right].
\end{aligned}$$

5.3 Low state solution: $\bar{T} \to \infty$

Taking the limit $\bar{T} \to \infty$, imposing the high state equilibrium outcomes under rational expectations, $x^{RE}_{H,t} = \pi^{RE}_{H,t} = 0$, and using the result that in the limit $\lim_{\bar{T}\to\infty} x^{RE}_{L,t} \to x_L$ and $\lim_{\bar{T}\to\infty} \pi^{RE}_{L,t} \to \pi_L$ for constants derived below, gives the aggregate demand curve

$$\begin{aligned}
x_t &= \sigma r_L + \frac{(1-\beta)(1-\delta)}{1-\beta(1-\delta)}x_L + \frac{\sigma(1-\delta)}{1-\beta(1-\delta)}\pi_L + \left[\frac{\sigma\beta(1-\delta)}{1-\beta(1-\delta)}r_L\right] \\
&\quad + \frac{1-\beta}{1-\beta\rho}\omega^x_{t|t-1} + \sigma\left[\frac{1-\beta}{1-\beta\rho} + \frac{\beta(1-\delta)}{1-\beta\rho(1-\delta)}\right]\omega^\pi_{t|t-1}
\end{aligned}$$

and the aggregate supply curve

$$\begin{aligned}
\pi_t &= \kappa\xi\beta\left[\frac{(1-\delta)}{1-\xi\beta(1-\delta)}x^{RE}_{L,t}\right] + (1-\xi)\,\beta\left[\frac{(1-\delta)}{1-\xi\beta(1-\delta)}\pi^{RE}_{L,t}\right] + \\
&\quad \kappa x_t + \left[\frac{\kappa\xi\beta}{1-\xi\beta\rho}\omega^x_{t|t-1}\right] + \left[\frac{(1-\xi)\beta}{1-\xi\beta\rho}\omega^\pi_{t|t-1}\right].
\end{aligned}$$

5.4 Rational expectations solution

To solve for the time-varying constants $x^{RE}_{L,t}$ and $\pi^{RE}_{L.t}$ proceed as follows. Under rational expectations, the high state has a time-invariant solution $\pi^{RE}_{H,t} = x^{RE}_{H,t} = 0$ and $R_t = r_H$ and

$\omega^x_{t|t-1} = \omega^\pi_{t|t-1} = 0$. Solving gives

$$\begin{aligned} x^{RE}_{L,t} &= (1-\beta)(1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}x^{RE}_{L,t} + \sigma(1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}\pi^{RE}_{L,t} \\ &\quad +\sigma\left(1+\beta(1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}\right)r_L \end{aligned}$$

and

$$\begin{aligned} \pi^{RE}_{L,t} &= kx^{RE}_{L,t} + \kappa\xi\beta\left((1-\delta)\frac{1-[\xi\beta(1-\delta)]^{\bar{T}+1-t}}{1-\xi\beta(1-\delta)}x^{RE}_{L,t}\right) + (1-\xi)\beta(1-\delta)\frac{1-[\xi\beta(1-\delta)]^{\bar{T}+1-t}}{1-\xi\beta(1-\delta)}\pi^{RE}_{L,t} \\ &= \left(\kappa + \kappa\xi\beta(1-\delta)\frac{1-[\xi\beta(1-\delta)]^{\bar{T}+1-t}}{1-\xi\beta(1-\delta)}\right)x^{RE}_{L,t} + (1-\xi)\beta(1-\delta)\frac{1-[\xi\beta(1-\delta)]^{\bar{T}+1-t}}{1-\xi\beta(1-\delta)}\pi^{RE}_{L,t} \end{aligned}$$

which can be solved for the time varying constants under rational expectations. These constants determine agents' prior beliefs about the consequences of a negative demand shock.

In the special case of the limit $\bar{T} \to \infty$, the rational expectations equilibrium under a two-state-Markov case satisfies

$$\begin{aligned} x^{RE}_{L,t} &= (1-\beta)(1-\delta)\frac{1}{1-\beta(1-\delta)}x^{RE}_{L,t} + \sigma(1-\delta)\frac{1}{1-\beta(1-\delta)}\pi^{RE}_{L,t} \\ &\quad +\sigma\left(1+\beta(1-\delta)\frac{1}{1-\beta(1-\delta)}\right)r_L \end{aligned}$$

and

$$\pi^{RE}_{L,t} = (1-\xi\beta(1-\delta))\left(\kappa + \kappa\xi\beta(1-\delta)\frac{1}{1-\xi\beta(1-\delta)}\right)x^{RE}_{L,t} + (1-\xi)\beta(1-\delta)\frac{1}{1-\xi\beta(1-\delta)}\pi^{RE}_{L,t}$$

which implies

$$\pi^{RE}_{L,t} = \frac{\kappa}{(1-\beta(1-\delta))}x^{RE}_{L,t}.$$

Substituting into the aggregate demand equation gives

$$x^{RE}_{L,t}\left(\frac{1}{\beta\delta-\beta+1}\left(\delta-\kappa\sigma-\beta\delta+\beta\delta^2+\kappa\sigma\delta\right)\right) = \sigma r_L$$

or

$$
\begin{aligned}
x_{L,t}^{RE} &= \frac{\sigma\left(1-\beta\left(1-\delta\right)\right)}{\left(\delta-\kappa\sigma-\beta\delta+\beta\delta^{2}+\kappa\sigma\delta\right)}r_{L}=x_{L} \\
\pi_{L,t}^{RE} &= \frac{\kappa\sigma}{\left(\delta-\kappa\sigma-\beta\delta+\beta\delta^{2}+\kappa\sigma\delta\right)}r_{L}=\pi_{L}
\end{aligned}
$$

which verifies the well-known solution of the model under rational expectations. As always, the denominator must be positive for a unique bounded rational expectations equilibrium.

Summary. In the low state aggregate demand and supply satisfy the solution

$$
\begin{aligned}
x_{t} &= x_{L}+\frac{1-\beta}{1-\beta\rho}\omega_{t|t-1}^{x}+\sigma\left[\frac{1-\beta}{1-\beta\rho}+\frac{\beta\left(1-\delta\right)}{1-\beta\rho\left(1-\delta\right)}\right]\omega_{t|t-1}^{\pi} \\
\pi_{t} &= \pi_{L}-\kappa x_{L}+\kappa x_{t}+\frac{\kappa\xi\beta}{1-\xi\beta\rho}\omega_{t|t-1}^{x}+\frac{(1-\xi)\beta}{1-\xi\beta\rho}\omega_{t|t-1}^{\pi}.
\end{aligned}
$$

In the high state we have

$$
\begin{aligned}
x_{t} &= -\sigma\left(R_{t}-r_{H}\right)+\frac{1-\beta}{1-\beta\rho}\omega_{t-1}^{x}+\sigma\frac{1-\beta}{1-\beta\rho}\omega_{t|t-1}^{\pi} \\
\pi_{t} &= kx_{t}+\frac{\kappa\xi\beta}{1-\xi\beta\rho}\omega_{t|t-1}^{x}+\frac{\left(1-\xi\right)\beta}{1-\xi\beta\rho}\omega_{t|t-1}^{\pi}.
\end{aligned}
$$

This is a complete statement of the household and firm behaviour in the low and high states. The central bank takes these decision rules as constraints in the optimal policy problem.

5.5 Optimal Policy: No Forward Guidance

The central bank minimizes the loss function

$$
L_{t}=E_{t}\sum_{T=t}^{\infty}\beta^{T-t}\frac{1}{2}\left(\pi_{T}^{2}+\lambda_{x}x_{T}^{2}\right) \tag{8}
$$

where $0<\beta<1$ and $\lambda_x>0$ determines the relative weight placed on inflation stabilization versus output gap stabilization. The Lagrangian for the optimal policy in this case is given by

$$\max_{\{\pi_t,x_t,R_t,\omega_t^\pi,\omega_t^x\}} E_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\begin{array}{c}\frac{1}{2}\left[\pi_t^2+\lambda_x x_t^2\right]\\ +\lambda_{1,t}\left(-\pi_t+\kappa x_t+\frac{(1-\xi)\beta}{1-\xi\beta\rho}\omega_{t|t-1}^\pi+\frac{\kappa\xi\beta}{1-\xi\beta\rho}\omega_{t|t-1}^x\right)\\ +\lambda_{2,t}\left(-x_t-\sigma(R_t-r_H)+\frac{1-\beta}{1-\beta\rho}\omega_{t|t-1}^x+\frac{\sigma(1-\beta)}{1-\beta\rho}\omega_{t|t-1}^\pi\right)\\ +\lambda_{3,t}\left(-\omega_{t+1|t}^\pi+\rho\omega_{t|t-1}^\pi+\rho\gamma\left(\pi_t-\omega_{t|t-1}^\pi\right)\right)\\ +\lambda_{4,t}\left(-\omega_{t+1|t}^x+\rho\omega_{t|t-1}^x+\rho\gamma\left(x_t-\omega_{t|t-1}^x\right)\right)\\ \lambda_{5,t}i_t\end{array}\right\}|S=H\right].$$

The first-order conditions are:

$$\begin{aligned}
\pi_t-\lambda_{1,t}+\rho\gamma\lambda_{3,t} &= 0\\
\lambda_x\left(x_t-x^*\right)+\lambda_{1,t}\kappa-\lambda_{2,t}+\lambda_{4,t}\rho\gamma &= 0\\
\frac{\kappa\xi\beta}{1-\xi\beta\rho}\beta E_t\lambda_{1,t+1}-\lambda_{4,t}+\beta\rho\left(1-\gamma\right)E_t\lambda_{4,t+1}+\beta\frac{1-\beta}{1-\beta\rho}E_t\lambda_{2,t+1} &= 0\\
\beta\frac{(1-\xi)\beta}{1-\xi\beta\rho}E_t\lambda_{1,t+1}+\beta\frac{\sigma(1-\beta)}{1-\beta\rho}E_t\lambda_{2,t+1}-\lambda_{3,t}+\beta\rho(1-\gamma)E_t\lambda_{3,t+1} &= 0\\
-\sigma\lambda_{2,t}+\lambda_{5,t} &= 0
\end{aligned}$$

with complementary slackness

$$\lambda_{5,t}\geq 0,\quad R_t\geq 0 \text{ and } \lambda_{5,t}R_t=0.$$

The remaining equations are

$$x_t = -\sigma(R_t-r_H)+\frac{1-\beta}{1-\beta\rho}\omega_{t|t-1}^x+\frac{\sigma(1-\beta)}{1-\beta\rho}\omega_{t|t-1}^\pi \quad (9)$$

$$\pi_t = \kappa x_t+\frac{(1-\xi)\beta}{1-\xi\beta\rho}\omega_{t|t-1}^\pi+\frac{\kappa\xi\beta}{1-\xi\beta\rho}\omega_{t|t-1}^x \quad (10)$$

$$\omega_{t+1|t}^\pi = \rho\omega_{t|t-1}^\pi+\rho\gamma(\pi_t-\omega_{t|t-1}^\pi) \quad (11)$$

$$\omega_{t+1|t}^x = \rho\omega_{t|t-1}^x+\rho\gamma(\pi_t-\omega_{t|t-1}^x) \quad (12)$$

This is Regime 3.

As discussed in the main text, when the natural rate reverts to the high state, the zero lower bound may continue to be a constraint because of drifting expectations. In that case, $R_t=0$ and demand is given by (9). The solution algorithm simply checks whether the central bank would choose a negative interest rate, and if so, sets it equal to zero. This is Regime 2. Finally, in the low state, the central bank must choose $R_t=0$. This is Regime 1. This completes the solution.

Note that agents' beliefs are state variables. The Lagrange multipliers are jump variables. This is the reverse of usual rational expectations commitment problem. For this reason, we do not need to solve for the Lagrange multipliers to determine equilibrium outcomes for output and inflation. The solution algorithm is then much simpler: start from the time of the shock and move forward in time. If economic conditions lead to a choice of negative interest rate, set $R_t = 0$. When economics conditions imply $R_t > 0$ (because of movements in the natural rate or beliefs), the interest rate must satisfy the above optimal policy problem.

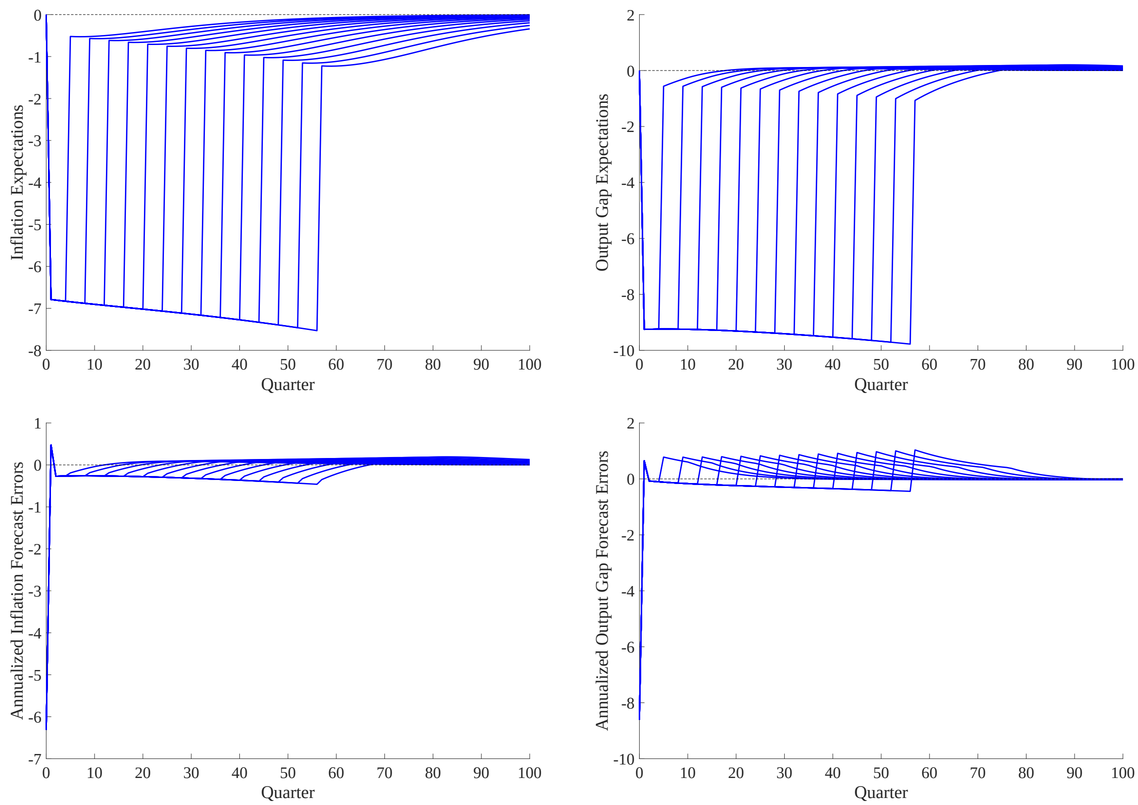
Of course, the Lagrange multipliers must satisfy the complementary slackness conditions. Therefore, we solve for the Lagrange multipliers as a check on our optimal solution. We use the same solution algorithm applied for the rational expectations commitment solution discussed in Section 8 of this appendix do these calculations.

Finally, note that for different parameter values two other distinct regimes are possible. The first is when the economy is in the low state but lift off occurs before the shock returns to the high state because policy has a highly stimulatory effect on expectations easing the the ZLB constraint. This regime is observed for modest negative demand shocks. The parameterization of the negative demand shocks that we consider are too large to allow for early lift off. The second occurs after liftoff from zero interest rate policy in response to a large negative demand shock. There can be a second bind of the zero lower bound after the initial lift off. This occurs when beliefs are highly unanchored and too much stimulus is put in the economy requiring large increases in the interest rate to restrain expectations. This policy can cause a second bind of the zero lower bound, usually lasting one quarter, followed by permanent lift off. Again, for the parameterizations we explore in the paper, this outcome does not occur. But we have explored such cases and they do not impact our main conclusions.

5.6 Dynamics of Forecast Errors

Figure 1 provides further details on the results of Section 3 on optimal policy without forward guidance. The figure shows the one-quarter ahead inflation and output gap expectations in the top panels and the associated forecast errors in the bottom panels. For visual clarity we plot realizations occurring four quarters apart. Focus on inflation, understanding that similar comments apply to the output gap. The first panel reveals a critical property of the model. As the duration of the shock lengthens inflation expectations decline monotonically. This reflects the endogenous drift component of expectations. Because of the ZLB the decline in inflation expectations triggered by the negative natural rate become self-confirming, producing the observed downward trend. This increasing pessimism is also reflected in inflation expectations at the time natural rate returns to the high state. The trend is reversed when the shock ends and the zero interest rate policy produces stimulus, driving up expectations and realized output and inflation.

Figure 1: Beliefs and Forecast Errors under Optimal Policy without Forward Guidance



Notes: Parameter values $g = 0.075$, $\rho = 0.985$, $\beta = 0.99$, $\sigma = 0.5$, $\delta = 0.1$, $\lambda_x = 0.05$, $\kappa = 0.02$, $r_L = -0.003$.

The forecast errors in the bottom panel tell the same story. After the initial large surprise, inflation forecast errors become progressively larger over time, though the effect is modest when compared to the initial surprise. The forecast errors lead to downward revisions in beliefs about inflation which cause the zero lower bound to be a constraint even after the shock dissipates. The larger the downward drift in expectations the more binding is the constraint of beliefs on monetary policy, increasing the time at the zero lower bound. Once the economy returns to the high state, the role played by output and output expectations in arresting the decline in inflation expectations is evident in the final panel.

6 Forward Guidance

Now suppose the central bank can commit to zero interest rate policy. There are three regimes that roughly correspond to the three regimes described in the last section. The first is defined by the natural rate at r_L and interest rates at the ZLB. The central bank makes a set of state-contingent credible promises. The second is defined by the natural rate reverting to r_H but interest rates remaining at zero, consistent with the announced commitment. The third is defined by the economy being in the high state, and the central bank using conventional interest rate policy.

Forward guidance policy is a set of state contingent promises to hold the policy rate at zero in the high state once the shock has reverted. We use the notation τ to denote the duration of the shock and $k_\tau \in \mathbb{N}^+$ to be the promised number of quarters of zero interest rates for a shock of duration τ. Therefore, the fully specified forward guidance policy of the central bank is $\{k_\tau\}_{\tau=1}^{\tau=\bar{T}}$, where $\bar{T}$ is the maximum duration of the shock, which is assumed known with certainty by all agents in the economy.

6.1 Regime 3

The third regime occurs in periods $t > \tau + k_\tau$ where τ is the duration of the shock and k_τ the period of zero interest rates attached to that state-contingent realization. The dynamics then coincide with regime 3 from the no forward guidance case and are written

$$\begin{aligned} x_t &= -\sigma\left(R_t - r_H\right) + \frac{1-\beta}{1-\beta\rho}\omega^x_{t|t-1} + \sigma\frac{1-\beta}{1-\beta\rho}\omega^\pi_{t|t-1} \\ \pi_t &= kx_t + \frac{\kappa\xi\beta}{1-\xi\beta\rho}\omega^x_{t|t-1} + \frac{(1-\xi)\beta}{1-\xi\beta\rho}\omega^\pi_{t|t-1}. \end{aligned}$$

In all simulations, we assume that policy is set optimally, so that the dynamics follow those described previously in 5.5.

6.2 Regime 2

During $\tau \leq t \leq \tau + k_\tau$ the natural rate is r_H but interest rates are zero so the zero interest rate policy is anticipated to continue beyond the shock. Starting with aggregate demand we have

$$\begin{aligned} x_t^{k_\tau(j)} &= \sigma r_H + E_t \sum_{T=t}^{t+k_\tau-j-1} \beta^{T-t}\left[(1-\beta)\,x_{T+1} - \sigma\left(0 - \pi_{T+1} - \beta r_H\right)\right] \\ &\quad + E_t \sum_{T=t+k_\tau-j}^{\infty} \beta^{T-t}\left[(1-\beta)\,x_{T+1} - \sigma\left(\beta R_{T+1} - \pi_{T+1} - \beta r_H\right)\right] \end{aligned}$$

where $x_t^{k_\tau(j)}$ denotes the output gap in a period under a promise of k_τ periods of zero interest rates with $k_\tau - j$ the remaining periods until lift off. Therefore, $j = 0$ corresponds to the time when the economy switches back to the high state, and the central bank implements

k_τ periods of additional zero interest rates. Using the general expression derived at the beginning of this appendix, we can evaluate expectations as

$$\begin{aligned} x_t^{k_\tau(j)} &= \sigma r_H + \left(1-(\beta\rho)^{k_\tau - j}\right)\left[\frac{1-\beta}{1-\beta\rho}\omega_{t|t-1}^x + \frac{\sigma}{1-\beta\rho}\omega_{t|t-1}^\pi\right] + \left(1-\beta^{k_\tau-j}\right)\frac{\sigma\beta}{1-\beta}r_H \\ &\quad + (\beta\rho)^{k_\tau-j}\left[\frac{1-\beta}{1-\beta\rho}\omega_{t|t-1}^x + \sigma\frac{1-\beta}{1-\beta\rho}\omega_{t|t-1}^\pi\right] \\ &= \sigma\left[\frac{1}{1-\beta} - \beta^{k_\tau-j}\frac{\beta}{1-\beta}\right]r_H + \frac{1-\beta}{1-\beta\rho}\omega_{t|t-1}^x + \left(1-\beta\,(\beta\rho)^{k_\tau-j}\right)\frac{\sigma}{1-\beta\rho}\omega_{t|t-1}^\pi. \end{aligned}$$

There are two sources of stimulus from the zero interest rate policy. The first term captures the reduction in nominal rates to zero relative to the real neutral rate for $k_\tau - j$ periods. Recall the standard one period effect is σr_H. The first term is simply the sum of this effect over $k_\tau - j$ periods. Of course, what matters is the effective real interest rate relative to the neutral rate. This is determined by the additional effect from inflation expectations in the final term.

The structure of optimal pricing decisions is unaffected so we have

$$\pi_t^{k_\tau(j)} = kx_t^{k_\tau(j)} + \frac{\kappa\xi\beta}{1-\xi\beta\rho}\omega_{t|t-1}^x + \frac{(1-\xi)\,\beta}{1-\xi\beta\rho}\omega_{t|t-1}^\pi.$$

6.3 Regime 1

Again, assume that in period $\bar{T}$ we return to the high state with probability 1. At that time, the central bank implements $k_{\bar{T}}$ periods of zero interest rate policy. Then

$$E_t\sum_{T=\bar{T}}^{\infty}\beta^{T-t}R_{T+1} = \frac{\beta^{k_{\bar{T}}}}{1-\beta}r_H + \frac{(\beta\rho)^{k_{\bar{T}}}}{1-\beta\rho}\omega_{t|t-1}^\pi$$

which generalizes, for any number of periods j before this date, to

$$E_{\bar{T}-j}\sum_{T=\bar{T}-j}^{\infty}\beta^{T-\bar{T}-j}R_{T+1} = \Psi_t^\rho\omega_{\bar{T}-j|\bar{T}-j-1}^\pi + \Psi_t^1 r^H$$

where for $\tilde{\rho} = \{\rho, 1\}$

$$\Psi_t^{\tilde{\rho}} = (1-\delta)\,\beta\Psi_{t+1}^{\tilde{\rho}} + \delta\frac{(\beta\tilde{\rho})^{k_t}}{1-\beta\tilde{\rho}}$$

for $j = 0, ..., (\bar{T}-1)$.

Aggregate demand then can be written

$$
\begin{aligned}
x_t &= E_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} - \sigma \left(R_T - \pi_{T+1} - r_T \right) \right] \\
&= -\sigma \left(R_t - r_t \right) + E_t \sum_{T=t}^{\bar{T}} \beta^{T-t} \left[(1-\beta) x_{T+1} - \sigma \left(\beta R_{T+1} - \pi_{T+1} - \beta r_{T+1} \right) \right] \\
&\quad + E_t \sum_{T=\bar{T}+1}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} - \sigma \left(\beta R_{T+1} - \pi_{T+1} - \beta r_{T+1} \right) \right] \\
&= \sigma r_L \\
&\quad + (1-\beta) \left[(1-\delta) \frac{1 - \left[\beta (1-\delta) \right]^{\bar{T}+1-t}}{1 - \beta (1-\delta)} x_{L,t}^{RE} \right] \\
&\quad + (1-\beta) \left[\frac{1 - \beta^{\bar{T}+1-t}}{1-\beta} - (1-\delta) \frac{1 - \left[\beta (1-\delta) \right]^{\bar{T}+1-t}}{1 - \beta (1-\delta)} \right] x_{H,t}^{RE} \\
&\quad + (1-\beta) \left[\frac{1 - (\beta\rho)^{\bar{T}+1-t}}{1-\beta\rho} \omega_{t|t-1}^{x} \right] \\
&\quad + \sigma \left[(1-\delta) \frac{1 - \left[\beta (1-\delta) \right]^{\bar{T}+1-t}}{1 - \beta (1-\delta)} \pi_{L,t}^{RE} + \left[\frac{1 - \beta^{\bar{T}+1-t}}{1-\beta} - (1-\delta) \frac{1 - \left[\beta (1-\delta) \right]^{\bar{T}+1-t}}{1 - \beta (1-\delta)} \right] \pi_{H,t}^{RE} \right] \\
&\quad + \sigma \left[+ \frac{1 - (\beta\rho)^{\bar{T}+1-t}}{1-\beta\rho} \omega_{t|t-1}^{\pi} \right] \\
&\quad + \sigma\beta \left[(1-\delta) \frac{1 - \left[\beta (1-\delta) \right]^{\bar{T}+1-t}}{1 - \beta (1-\delta)} r_L + \left[\frac{1 - \beta^{\bar{T}+1-t}}{1-\beta} - (1-\delta) \frac{1 - \left[\beta (1-\delta) \right]^{\bar{T}+1-t}}{1 - \beta (1-\delta)} \right] r_H \right] \\
&\quad - \sigma\beta \left[\Psi_t^{\rho} \omega_{t|t-1}^{R} + \Psi_t^{1} r_H \right] \\
&\quad + (1-\beta) \left[\frac{\beta^{\bar{T}+1-t}}{1-\beta} x_t^H + \frac{(\beta\rho)^{\bar{T}+1-t}}{1-\beta\rho} \omega_{t|t-1}^{x} \right] \\
&\quad + \sigma \left[\frac{\beta^{\bar{T}+1-t}}{1-\beta} \pi_t^H + \frac{(\beta\rho)^{\bar{T}+1-t}}{1-\beta\rho} \omega_{t|t-1}^{\pi} \right] \\
&\quad + \sigma\beta \left[\frac{\beta^{\bar{T}+1-t}}{1-\beta} r_H \right]
\end{aligned}
$$

or simplifying (using the fact that the RE equilibrium in the high state implies inflation and the output gap are zero) gives

$$\begin{aligned} x_t^L &= \sigma r_L \\ &+ (1-\beta)\left[(1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}x_{L,t}^{RE} + \frac{1}{1-\beta\rho}\omega_{t|t-1}^{x}\right] \\ &+\sigma\left[(1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}\pi_{L,t}^{RE} + \frac{1}{1-\beta\rho}\omega_{t|t-1}^{\pi}\right] \\ &+\sigma\beta\left[(1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}r_L + \left[\frac{1}{1-\beta} - (1-\delta)\frac{1-[\beta(1-\delta)]^{\bar{T}+1-t}}{1-\beta(1-\delta)}\right]r_H\right] \\ &-\sigma\beta\left[\Psi_t^{\rho}\omega_{t|t-1}^{\pi} + \Psi_t^{1}r_H\right]. \end{aligned}$$

Taking the limit $\bar{T} \to \infty$ and using the solution for the rational expectations equilibrium value x_L provides the expression in the main text.

7 Optimal Policy

This section describes how we solve optimal policy for the learning model.

7.1 The Policy Problem

The central bank minimizes the loss function

$$L_t = E_t \sum_{T=t}^{\infty} \beta^{T-t}\frac{1}{2}\left(\pi_T^2 + \lambda_x x_T^2\right) \tag{13}$$

subject to the dynamics described in Regimes 1-3 in the previous section where $0 < \beta < 1$ and $\lambda_x > 0$ determines the relative weight placed on inflation stabilization versus output gap stabilization. There is not an analytic solution to this non-linear problem. Therefore, following Eggertsson and Woodford (2003) we use a numerical approach to approximate optimal policy.

We wish to approximate the optimal policy solution for the case where the low state could persist indefinitely. Following Eggertsson and Woodford (2003), we accomplish this by noting that if we choose $\bar{T}$ large enough, that the effect of any expectations of the shock ending with certainty become vanishingly small. This allows us to pick a finite $\bar{T}$, propose different policies, and numerically search for the optimum. However, for even moderately large values of $\bar{T}$, the number of possible state contingent policies we need to check becomes computationally infeasible. Therefore, we restrict our search to a grid of monotonic profiles that are either increasing, flat, or decreasing with the duration of the shock, and therefore consistent with both 'back-loaded' and 'front-loaded' policies motivated by the simple analytics of Section 5.

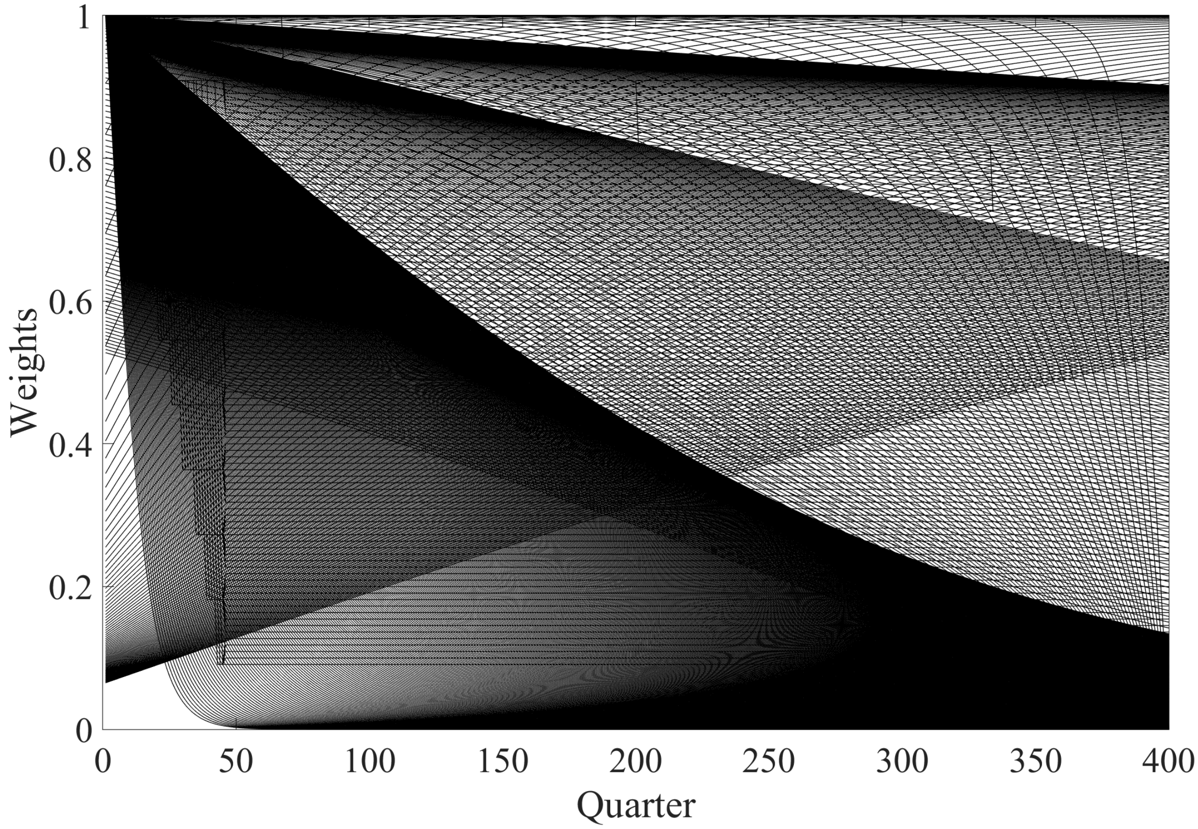
To construct the majority of the policy profiles that we consider we start with a set of flat promise profiles. That is we consider 16 policies that promise a constant period of zero interest policy for all durations of the shock, starting with 5 quarters and ending with 20 quarters (in practice we have searched from 1 quarter to 35 quarters for our preferred parameterization). We then twist these flat profiles to generate rising and falling profiles of promises with varying degrees of concavity and convexity.

The twisted profiles are constructed by using the hyperbolic tangent function to generate weights which are applied to the flat profiles. Figure 2 shows the set of weights we consider in our grid search. There are 790 smooth weight-profiles with shapes that clearly span the space of rising and falling profiles. Figure 3 illustrates the process. The first panel shows one particular tangent function. The second panel shows one possible section of the function that can be pulled to generate weights. The third and fourth panels show how we re-scale this segment to deliver weights for a declining profile of promises. We create $\{k_\tau\}_{\tau=1}^{\tau=\bar{T}}$ by using these profiles as weighting functions. In other words, we multiply a flat or constant profile where, for example, $\{\bar{k} = k_\tau\}_{\tau=1}^{\tau=\bar{T}}$ and $\bar{k} = 15$ by one of the profiles shown in Figure 2 and round the results to the nearest whole number. For a front-loaded profile (with weights equal to 1 and progressively declining), this results in a state contingent policy such as $\{15, 14, 14, 13, ..., 0, 0, 0\}$. For a back-loaded profile (with weights initially equal to zero then rising), this results in a state contingent policy such as $\{0, 0, 0, 1, ..., 14, 14, 15\}$. In addition to these smooth profiles we consider policies that progressively and at varying rates reduce promises according to a step function over the first 45 quarters and then flat thereafter.[3] Once these weights are applied to the sixteen flat promise profiles we get a set of over 15000 forward guidance policies. Finally, we consider a set of calendar-based policies which hold interest rates at zero for a fixed period after the shock. We evaluate all calendar-based policies from one through to thirty six quarters. In this way we approximate the space of all possible monotonic profiles.

Once a profile is constructed, we then simulate the economy for all realizations of $\tau \leq \bar{T}$ and calculate the central bank's loss by computing the time-zero expectation of those paths by weighting the actual outcomes by the probability they occur and the central bank's discount factor, β. Specifically, to simulate the economy for a given forward guidance policy $\{k_\tau\}_{\tau=1}^{\tau=\bar{T}}$ and τ, we start in Regime 1. Output and inflation given the policy evolve according

[3] The latter exploring whether it is possible to obtain near term gains by making larger promises in states of the world that are unlikely to occur as is optimal in the rational model.

Figure 2: Numerically Solving for Optimal Forward Guidance Policy



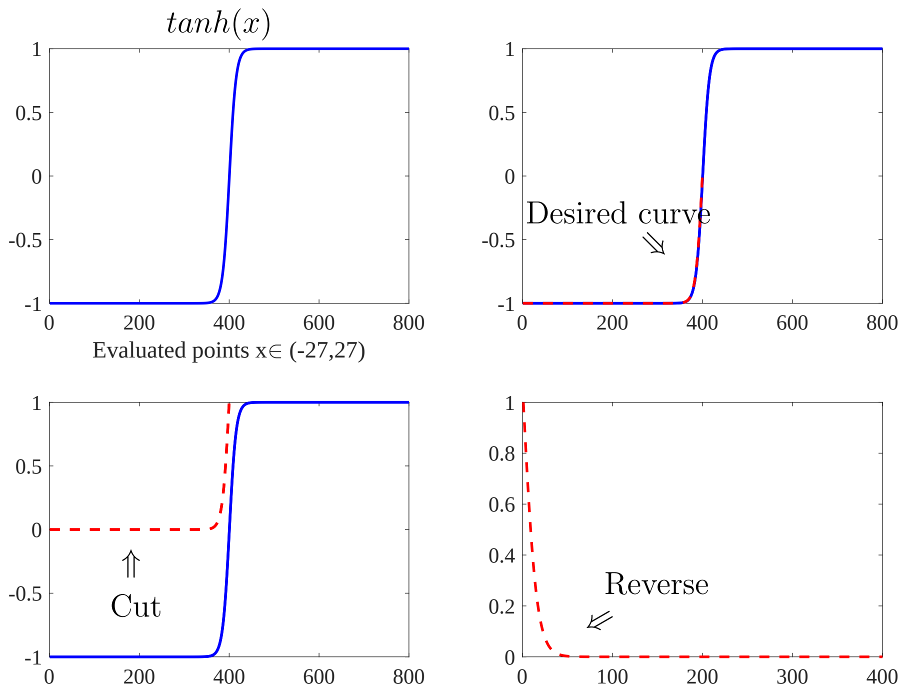
Notes: General types of policy profiles considered when searching for optimal policy.

to the following system of equations:

$$
\begin{aligned}
\pi_t^L &= kx_t^L + (\pi_L - \kappa x_L + \frac{\kappa\xi\beta}{1-\xi\beta\rho}\omega^x_{t|t-1} + \frac{(1-\xi)\beta}{1-\xi\beta\rho}\omega^\pi_{t|t-1} \\
x_t^L &= x^{RE}_{L,t} + \frac{(1-\beta)}{1-\beta\rho}\omega^x_{t|t-1} + \sigma\left(\frac{1}{1-\beta\rho} - \beta\Psi^\rho_t\right)\omega^\pi_{t|t-1} \\
&\quad + \sigma\beta\left[\frac{1}{1-\beta} - \frac{(1-\delta)}{1-\beta(1-\delta)} - \Psi^1_t\right] r_H \\
\omega^\pi_{t+1|t} &= \begin{cases} \rho\omega^\pi_{t|t-1} + \rho\gamma(\pi^L_t - \omega^\pi_{t|t-1}) & \text{if } t = 1 \\ \rho\omega^\pi_{t|t-1} + \rho\gamma(\pi^L_t - \pi_L - \omega^\pi_{t|t-1}) & \text{if } t > 1 \end{cases} \\
\omega^x_{t+1|t} &= \begin{cases} \rho\omega^x_{t|t-1} + \rho\gamma(x^L_t - \omega^x_{t|t-1}) & \text{if } t = 1 \\ \rho\omega^x_{t|t-1} + \rho\gamma(x^L_t - x_L - \omega^x_{t|t-1}) & \text{if } t > 1 \end{cases}
\end{aligned}
$$

We simulate the economy forward in the low state using the above equations for $t = 1, ..., \tau$ periods. Then, we enter Regime 2, where the promised policy is delivered. Regime 2 lasts

Figure 3: Constructing the Grid



Notes: This figure shows how we construct the weights for our grid search for optimal policy. We scale the hyperbolic tangent and then use segments of it to construct different possible policy profiles.

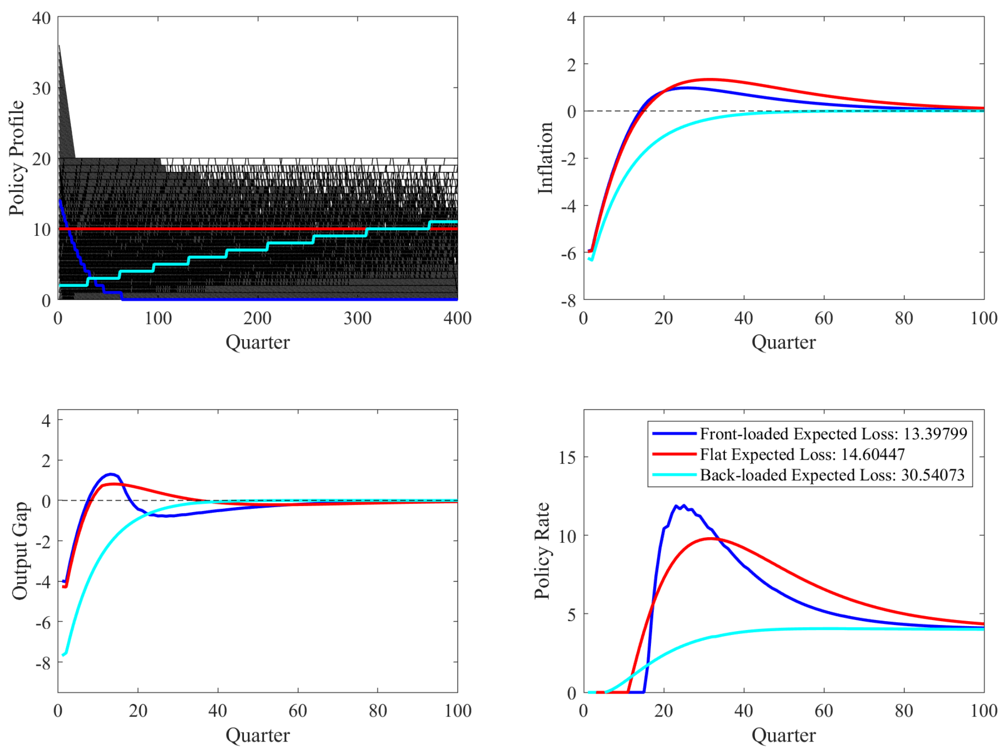
for k_τ periods and evolves as

$$
\begin{aligned}
\pi_t^{k_\tau(j)} &= kx_t^{k_\tau(j)} + \frac{\kappa\xi\beta}{1-\xi\beta\rho}\omega_{t|t-1}^x + \frac{(1-\xi)\beta}{1-\xi\beta\rho}\omega_{t|t-1}^\pi \\
x_t^{k_\tau(j)} &= \sigma\left[\frac{1}{1-\beta} - \beta^{k_\tau - j}\frac{\beta}{1-\beta}\right] r_H + \frac{1-\beta}{1-\beta\rho}\omega_{t|t-1}^x + \left(1-\beta\left(\beta\rho\right)^{k_\tau - j}\right)\frac{\sigma}{1-\beta\rho}\omega_{t|t-1}^\pi \\
\omega_{t+1|t}^\pi &= \rho\omega_{t|t-1}^\pi + \rho\gamma(\pi_t^L - \omega_{t|t-1}^\pi) \\
\omega_{t+1|t}^x &= \rho\omega_{t|t-1}^x + \rho\gamma(x_t^L - \omega_{t|t-1}^x)
\end{aligned}
$$

Finally, we lift off into Regime 3 and economy evolves according the optimal policy problem in the high state without forward guidance, as described in Section 5.5 of this appendix. We repeat this simulation for all $\tau = 1, ..., \bar{T}$ and use all possible paths for π_t and x_t to compute the central bank's expected discounted loss at time zero using the Markov probabilities.

To illustrate the outcome of this simulation, Figure 4 shows the expected path of the economy for three example policy profiles: one back-loaded policy resembling the character of optimal policy in a rational expectations model, one constant or flat promise, and one front-loaded policy. The 15,000+ profiles we consider in a standard search is shown in the top right with the specific profiles considered highlighted. Once the optimal policy is found from this grid search, we also explore perturbations of the policy profile by adding and removing a single promises along the profile to see if there are obvious gains from making small changes. If gains are found, we perturb the new profile by adding or removing single promises again until no further gains are found. In practice, this additional step often does find modest differences in the profiles that improve welfare. Welfare gains from these changes, though, are typically empirically irrelevant resulting in improvements in the neighborhood of 1/10,000 of a percentage point of total expected welfare.

Figure 4: Numerically Solving for Optimal Forward Guidance Policy: Exmaples



Notes: General types of policy profiles considered when searching for optimal policy and their expected impact on output, inflation, and the policy rate.

7.2 Checking the accuracy of the approximation for a given $\bar{T}$

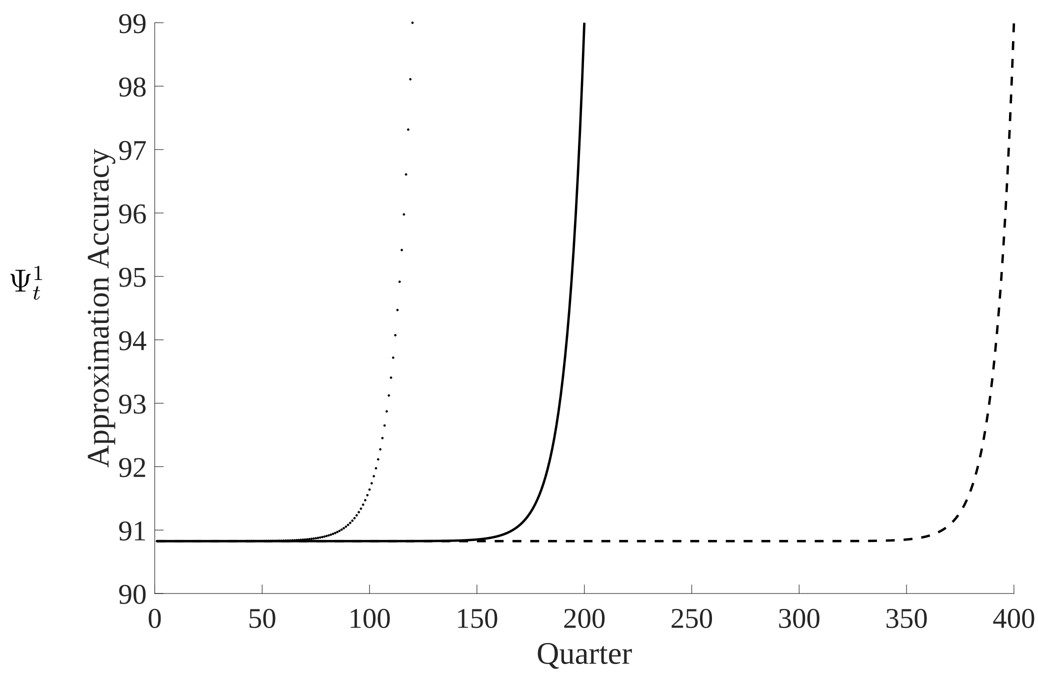
To construct a solution for the optimal forward guidance problem we assume that the economy returns to the high state with probability one in some future period $\bar{T}$. Without this assumption we would have to compute the forward guidance promises, encoded in the recursions

$$\Psi_t^{\tilde{\rho}} = (1-\delta)\,\beta\Psi_{t+1}^{\tilde{\rho}} + \delta\frac{(\beta\tilde{\rho})^{k_t}}{1-\beta\tilde{\rho}}$$

for $\tilde{\rho} = \{\rho, 1\}$, into the infinite future. The assumption of exiting the low state with certainty truncates the infinite sum to one with a finite number of terms equal to $\bar{T}$. We then choose $\bar{T}$ large enough such that our solution for the first 50 quarters after the shock always remains the same, regardless of the exact choice of $\bar{T}$. In this way, we generate an accurate approximation to the true two-state Markov process for the most probable realizations of uncertainty.

To illustrate how large $\bar{T}$ must be, Figure 5 plots Ψ_t^1 for $\bar{T} = 120, 200$, and 400 under the assumption of a 'flat promise' in all future periods of one-quarter of zero interest rate policy, $k_\tau = 1$ for all τ. Note that Ψ_t^1 is roughly constant for all realizations of uncertainty that are at least 50 periods away from the known exit point. Because this property is determined by the eigenvalue in the difference equation, it also holds for our optimal promises.

Figure 5: Approximation

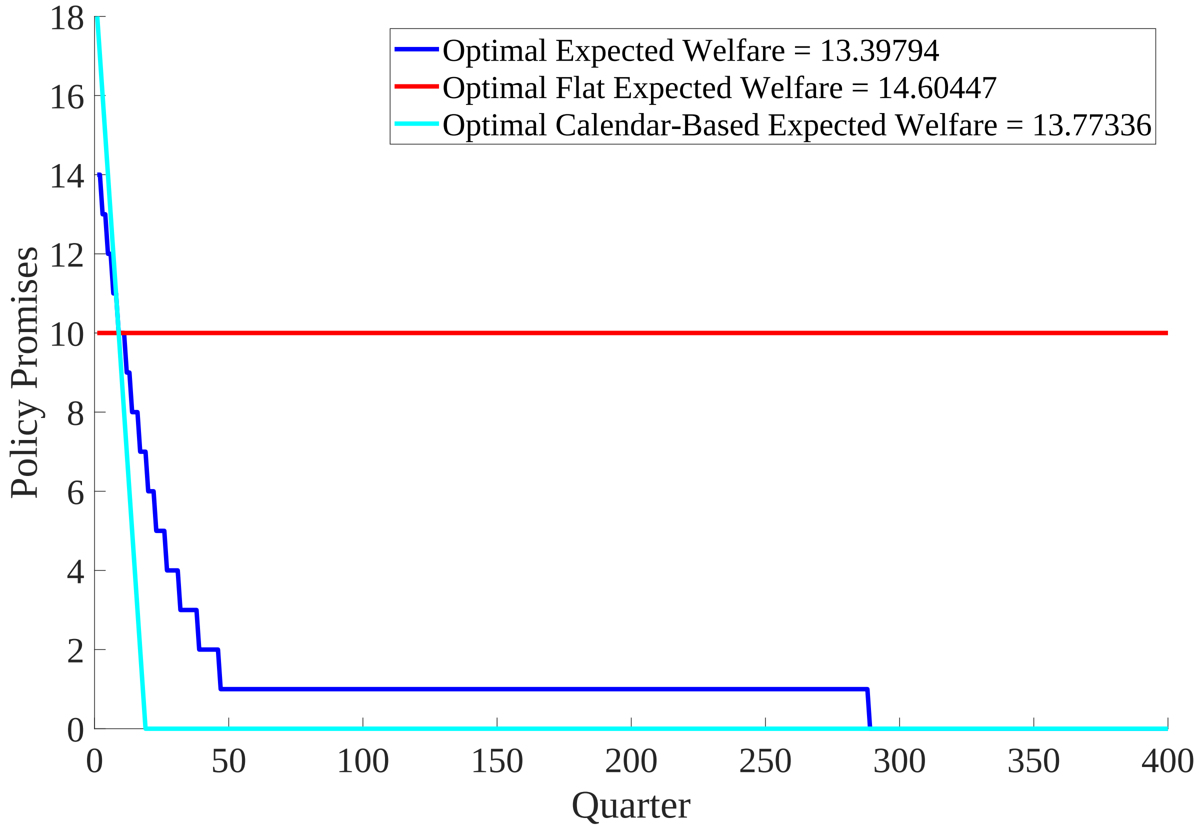


Notes: Parameter values $g = 0.075$, $\beta = 0.99$, $\sigma = 0.5$, $\delta = 0.1$, $\lambda_x = 0.05$, $\kappa = 0.02$, $r_L = -0.003$.

7.3 Numerical search for optimal policy

To numerically find optimal policy, we set $\bar{T} = 400$ and put forward 15,010 possible policy profiles to evaluate. We calculate welfare for each profile. Once the best profile is found, we check perturbations around the profile to find if there are obvious gains to be had by altering the promise in any way. Figure 6 shows optimal state-contingent promise from the numerical search compared to the best 'flat' or constant promise and the best calendar-based forward guidance policy promise for our preferred calibration of the model. The expected welfare of each policy is shown in the legend.

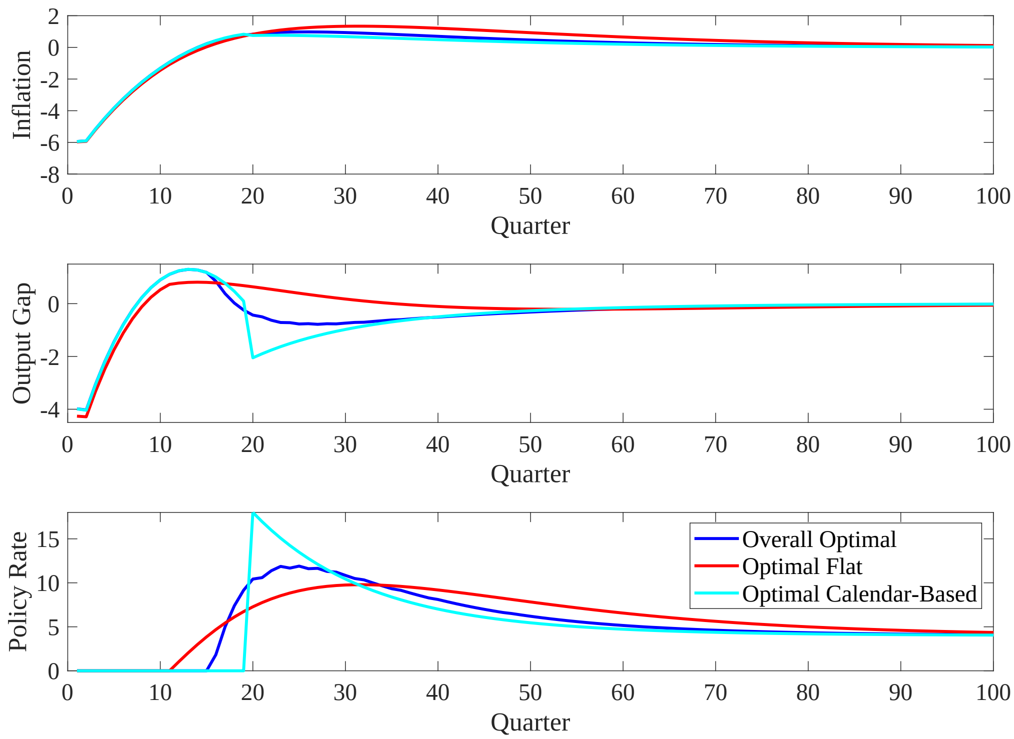
Figure 6: Numerically Solving for Optimal Forward Guidance Policy



Notes: Optimal policy compared to two other possible state-contingent policy promises. Parameter values $g = 0.075$, $\beta = 0.99$, $\sigma = 0.5$, $\delta = 0.1$, $\lambda_x = 0.05$, $\kappa = 0.02$, $r_L = -0.003$.

Figure 7 shows the time zero expected paths of inflation, output, and the policy rate under three different policies reported in Figure 6. The figure illustrates the sense in which calendar-based policies are able to well-approximate the overall optimal policy. Because the central bank weights expected misses of its inflation objective much more highly than it weights misses on output, the central bank does not lose much in expectation by being more aggressive up front. The better inflation outcomes achieved by this more aggressive, more front-loaded policy nearly offset all the additional losses from the recession that the central bank must engineer to restrain expectations when the shock resolves. This leads to expected losses that are nearly as favorable as those observed under the overall optimum.

Figure 7: Comparing different types of forward guidance policy



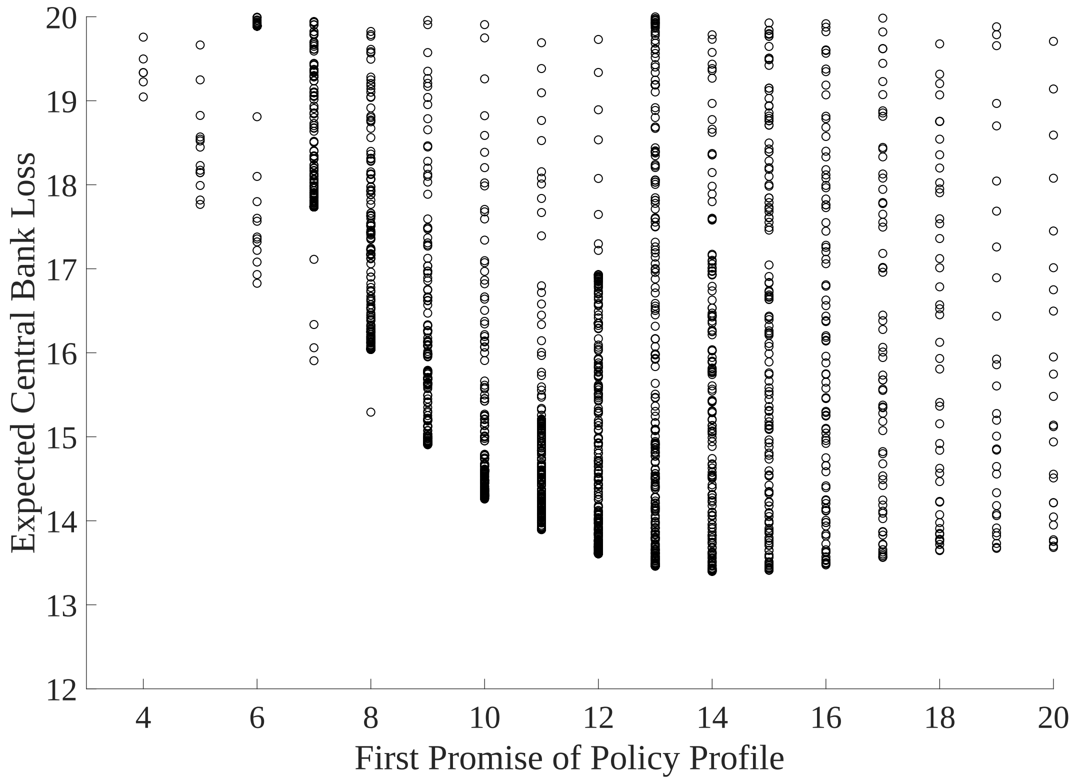
Notes: Parameter values $g = 0.075$, $\beta = 0.99$, $\sigma = 0.5$, $\delta = 0.1$, $\lambda_x = 0.05$, $\kappa = 0.02$, $r_L = -0.003$.

Finally, to illustrate that the policy problem has curvature such that there is a well-defined optimum, Figure 8 shows the expected central bank loss from the grid search for the best policies we found plotted against the duration of the first promise for a shock lasting a single quarter. Optimal policy with rational expectations usually requires the central bank to make no, or only a single quarter of promised zero interest rate policy for a shock lasting one quarter. Policies with small initial promises are clearly shown to have lower expected welfare than policies with large initial promises. In addition, there is clear curvature with expected welfare maximized with promises starting at 14 quarters.

7.4 Reneging on a policy promise

We also explore the possibility that the central bank reneges on a promised path. This is accomplished by simply transitioning directly from Regime 1 to Regime 3, or ending Regime 2 prematurely to enter Regime 3. We do not consider early exit from Regime 1, though

Figure 8: Numerically Solving for Optimal Forward Guidance Policy



Notes: This figure shows the expected welfare of the proposed state-contingent policies plotted against the size of the first promise of each profile for a shock lasting one quarter. The graph illustrates that there is curvature to the minimization problem for the grid of policies that we have searched. The majority of policies result in expected central bank loss well above 20 and are not shown for clarity. Parameter values $g = 0.075$, $\beta = 0.99$, $\sigma = 0.5$, $\delta = 0.1$, $\lambda_x = 0.05$, $\kappa = 0.02$, $r_L = -0.003$.

it is feasible, simply because the rational expectations component of beliefs requires the assumption of $R_t = 0$. The renege solution is calculated simply by checking whether the central bank prefers to set interest above zero once the shock reverts, i.e., given current private sector expectations whether optimal policy in Regime 3 requires a non-zero interest rate.

8 Optimal policy in RE, BNK, and THANK

Our solution method is based on Eggertsson and Woodford (2003). We also use the same method to calculate optimal policy for the Gabaix (2020) behavioral New Keynesian model and the THANK model of Bilbiie (2018). This same solution method is also discussed at length in Eggertsson, Egiev, Lin, Platzer, and Riva (2021). The solution method considers three regimes

1. Regime 1: the real rate of interest is in the low state and state-contingent forward

guidance promises are announced

2. Regime 2: the real rate of interest is in the high state and $R_t = 0$ delivering the promised policy

3. Regime 3: the real rate of interest is in the high state and the economy is governed by optimal commitment policy

The RE solution requires one to account for all three regimes. The solution method is backward induction. The economy is governed by

$$\pi_t = M_f \beta E_t \pi_{t+1} + \kappa x_t \tag{14}$$
$$x_t = M E_t x_{t+1} - \sigma N (R_t - E_t \pi_{t+1} - r_t^n) \tag{15}$$

where M, M_f, and N are added so that we capture a range of modeling assumptions. Different choices of the parameters M, M_f, and N deliver various models that have been proposed to solve the forward guidance puzzle. For example, Del Negro, Giannoni, and Patterson (2012), Angeletos and Lian (2018), Gabaix (2020), Bilbiie (2018), McKay, Nakamura, and Steinsson (2017) and Farhi and Werning (2019) all have this structure to a first order approximation.

The central bank takes these relationships as given and commits to a path for policy that minimizes (13). The Lagrangian that summarizes the problem is

$$\mathcal{L}_t = E_t \sum_{T=t}^{\infty} \beta^{T-t} \frac{1}{2} \left(\pi_T^2 + \lambda x_T^2) \right) \tag{16}$$
$$+ \sum_{T=t}^{\infty} \beta^{T-t} \phi_{1,T} \left(-x_T + M x_{T+1} - \sigma N (i_T - \pi_{T+1} - r_T^n) \right) \tag{17}$$
$$+ \sum_{T=t}^{\infty} \beta^{T-t} \phi_{2,T} \left(-\pi_T + M_f \beta \pi_{T+1} + \kappa x_t \right) \tag{18}$$

The first order conditions are

$$\pi_t - \phi_{2,t} + M_f \phi_{2,t-1} + \beta^{-1} \sigma N \phi_{1,t-1} = 0 \tag{19}$$
$$\lambda x_t + \kappa \phi_{2,t} - \phi_{1,t} + \beta^{-1} M \phi_{1,t-1} = 0 \tag{20}$$
$$\tag{21}$$

with complementary slackness

$$\phi_{1,t} \geq 0, \quad i_t \geq 0 \text{ and } \phi_{1,t} i_t = 0.$$

Regime 3: To find the solution, we begin in Regime 3 by stacking the first order conditions and the IS and Phillips curve and putting in matrix form. Define the state vector as:

$$\begin{bmatrix} \pi_t \\ x_t \\ i_t \\ \phi_{1,t-1} \\ \phi_{2,t-1} \\ r_t^n \end{bmatrix}$$

so the system can be written

$$A E_t Z_{t+1} = B Z_t$$

where

$$\begin{bmatrix} M_f\beta & 0 & 0 & 0 & 0 & 0 \\ \sigma N & M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & \kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} E_t \begin{bmatrix} \pi_{t+1} \\ x_{t+1} \\ i_{t+1} \\ \phi_{1,t} \\ \phi_{2,t} \\ r_{t+1}^n \end{bmatrix} = \begin{bmatrix} 1 & -\kappa & 0 & 0 & 0 & 0 \\ 0 & 1 & \sigma N & 0 & 0 & -\sigma N \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \beta^{-1}\sigma N & -M_f & 0 \\ 0 & \lambda_x & 0 & \beta^{-1} M & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \\ i_t \\ \phi_{1,t-1} \\ \phi_{2,t-1} \\ r_t^n \end{bmatrix}$$

We have the standard solution:

$$\begin{aligned} P_t &= \Omega^0 P_{t-1} + \Theta^0 \\ Z_t &= \Lambda^0 P_{t-1} + \Phi^0 \end{aligned} \tag{22}$$

which must hold for all t in Regime 3 and where Θ^0 and Φ^0 are included for generality but are zero in this exercise.

Regime 2: In Regime 2, interest rates are held at zero for a known duration. To enforce the zero interest rate policy, we set $\phi_{1,t} = 0$ and $i_t = 0$. This requires that we change $A(3,4) = 0$ and $A(3,3) = B(3,3) = 1$ in the above matrices. Let the duration of the policy be denoted as $k_\tau \in \mathbb{N}^+$, where τ represents the state of nature where such a policy is delivered. The solution path is then defined recursively starting from

$$
\begin{aligned}
Z_t &= \Lambda^0 P_{t-1} + \Theta^0 \\
P_t &= \Omega^0 P_{t-1} + \Phi^0
\end{aligned}
$$

and moving backwards in time, we have

$$
\begin{aligned}
Z_{t+j} &= \Lambda^j P_{t+j-1} + \Theta^j \\
P_{t+j} &= \Omega^j P_{t+j-1} + \Phi^j
\end{aligned}
$$

where $j = 0, ..., k_\tau$.

The values of Λ^j, Ω^j, Θ^j, and Φ^j are computed as follows:

$$
\begin{pmatrix} P_t \\ Z_t \end{pmatrix} = \begin{pmatrix} AA_2 & BB_2 \\ CC_2 & DD_2 \end{pmatrix} \begin{pmatrix} P_{t-1} \\ E_t Z_{t+1} \end{pmatrix} + \begin{pmatrix} M \\ V \end{pmatrix}
$$

where AA_2, BB_2, CC_2, and DD_2 are the appropriate submatrices that are constructed by rearranging A and B. Then, starting from $j = 1, ..., k_\tau$, we can recover the time-varying solution from

$$
\begin{aligned}
\Omega^j &= \left(I - BB_2 \Lambda^{j-1}\right)^{-1} AA_2 \\
\Lambda^j &= CC_2 + DD_2 \Lambda^{j-1} \Omega^j \\
\Phi^j &= \left(I - BB_2 \Lambda^{j-1}\right)^{-1} \left(BB_2 \Theta^{j-1} + M_2\right) \\
\Theta^j &= DD_2 \Lambda^{j-1} \Phi^j + DD_2 \Theta^{j-1} + V_2.
\end{aligned}
$$

Regime 1: In regime 1, we need to build the Markov shock process and the state contingent forward guidance announcement into agents' beliefs. First, assume that there is an agreed upon maximum duration of the shock $\bar{T}$. Then, we suppose that the central bank makes a promise for each possible realization of the shock $1, ..., \bar{T} - 1$ such that we have $\{k_\tau\}_{\tau=1}^{\tau=\bar{T}-1}$. For each k_τ, we have a solution from Regime 2: Λ^{k_τ}, Ω^{k_τ}, Θ^{k_τ}, and Φ^{k_τ}. Using these solutions and the Markov shock, the economy in each period evolves as

$$\left(\begin{bmatrix} (1-\delta)M_f\beta & 0 & 0 & 0 & 0 & 0 \\ (1-\delta)\sigma N & (1-\delta)M & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & \kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \delta M_f\beta & 0 & 0_{1\times 4} \\ \delta\sigma N & \delta M & 0_{1\times 4} \\ 0_{4\times 1} & 0_{4\times 1} & 0_{4\times 4} \end{bmatrix} \odot \begin{bmatrix} \Lambda^{k_\tau} & 0_{2\times 4} \\ 0_{4\times 2} & 0_{4\times 4} \end{bmatrix}\right) E_t \begin{bmatrix} \pi_{t+1} \\ x_{t+1} \\ i_{t+1} \\ \phi_{1,t} \\ \phi_{2,t} \\ r^n_{t+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\kappa & 0 & 0 & 0 & 0 \\ 0 & 1 & \sigma N & 0 & 0 & -\sigma N \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & \beta^{-1}\sigma N & -M_f & 0 \\ 0 & \lambda_x & 0 & \beta^{-1}M & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \pi_t \\ x_t \\ i_t \\ \phi_{1,t-1} \\ \phi_{2,t-1} \\ r_t \end{bmatrix}$$

where $\odot$ is the Hadamard product or element-by-element multiplication between the matrices. The above can then be written as

$$\begin{pmatrix} P_t \\ Z_t \end{pmatrix} = \begin{pmatrix} AA_{k_\tau} & BB_{k_\tau} \\ CC_{k_\tau} & DD_{k_\tau} \end{pmatrix} \begin{pmatrix} P_{t-1} \\ E_t Z_{t+1} \end{pmatrix} + \begin{pmatrix} M_{k_\tau} \\ V_{k_\tau} \end{pmatrix}$$

Then, starting with $\bar{T}-1$, where lift off occurs with probability one in the next period, we have

$$\Lambda^{k_{\bar{T}-j}} = \begin{cases} AA_3 & \text{if } k_{\bar{T}-1} = 0 \\ AA_2 & \text{if } k_{\bar{T}-1} > 0 \end{cases}$$

$$\Omega^{k_{\bar{T}-j}} = \begin{cases} CC_3 & \text{if } k_{\bar{T}-1} = 0 \\ CC_2 & \text{if } k_{\bar{T}-1} > 0 \end{cases}$$

$$\Theta^{k_{\bar{T}-j}} = \begin{cases} M_3 & \text{if } k_{\bar{T}-1} = 0 \\ M_2 & \text{if } k_{\bar{T}-1} > 0 \end{cases}$$

$$\Phi^{k_{\bar{T}-j}} = \begin{cases} V_3 & \text{if } k_{\bar{T}-1} = 0 \\ V_2 & \text{if } k_{\bar{T}-1} > 0 \end{cases}$$

where underscore 3 and 2 on the matrices denotes whether we enter Regime 3 or Regime 2 when $j = 0$ based on what promise the central bank has made. For $j = 1, ..., (\bar{T}-1)$, we have

$$\begin{aligned}
\Omega^{k_{\bar{T}-j}} &= \left(I - BB_{k_{\bar{T}-j}} \Lambda^{k_{\bar{T}-J+1}}\right)^{-1} AA_{k_{\bar{T}-j}} \\
\Lambda^{k_{\bar{T}-j}} &= CC_{k_{\bar{T}-j}} + DD_{k_{\bar{T}-j}} \Lambda^{k_{\bar{T}-J+1}} \Omega^{k_{\bar{T}-J}} \\
\Phi^{k_{\bar{T}-j}} &= \left(I - BB_{k_{\bar{T}-j}} \Lambda^{k_{\bar{T}-J+1}}\right)^{-1} \left(BB_2 \Theta^{k_{\bar{T}-J+1}} + M_{k_{\bar{T}-j}}\right) \\
\Theta^{k_{\bar{T}-j}} &= DD_{k_{\bar{T}-j}} \Lambda^{k_{\bar{T}-J+1}} \Phi^{k_{\bar{T}-J}} + DD_{k_{\bar{T}-J}} \Theta^{k_{\bar{T}-J+1}} + V_{k_{\bar{T}-j}}.
\end{aligned}$$

Numerical Solution: The solution to the problem that incorporates the three regimes is found numerically. It is implemented in the code OP_RE_and_Alternatives.m. The solution uses a guess and check method. First, a maximum duration is chosen. For all simulations, we choose 120, i.e., the real rate reverts back to the high state with certainty in period 121. Then, a constant promise is made, where the same amount of forward guidance is given for each realization of the shock. That promise is then used to recursively solve for the Regimes 1 and 2. We simulate the economy for each possible realization of the shock under this promise. We search for the smallest such promise that guarantees liftoff of interest rates at the end of the promised policy.

Once the smallest constant promise is found, we refine the promises. We do a forward sweep by starting in period one and lowering the promise there while leaving all other promises fixed. For each lower promise in period one, we check each realization of uncertainty to make sure that liftoff is still achieved. Once the lowest such promise is found in period one, we do the same for period two holding period one's new promise fixed. We repeat until promise 120. This recovers the optimal monotonically increasing policy.

9 Simple Analytics

We have the model of the form

$$c_t(i) = E_t^i \sum_{T=t}^{\infty} (m\beta)^{T-t} \left[(1-\beta) x_T - \beta\sigma \left(R_T - \pi_{T+1} - \bar{r}\right)\right]$$

where m denotes additional discounting as proposed by Gabaix. Proceed assuming that prices are fixed so that conditional expectations of future inflation are equal to zero. The model is closed with a goods market-clearing condition

$$x_t = \int c_t(i)\, di$$

an assumption on expectations, and an assumption about monetary policy. Note that under both rational expectations and learning we can write the optimal consumption demand as

$$c_t\left(i\right)=E_t^i\sum_{T=t}^{\infty}\left(m\beta\right)^{T-t}\left[\left(1-\beta\right)x_T-\beta\sigma\pi_{T+1}\right]-\beta\sigma E_t^i\sum_{T=t}^{\infty}\left(m\beta\right)^{T-t}\left(R_T-\bar{r}\right)$$

where the second term is the pure partial equilibrium effect of interest rate changes. That is, the effect of interest rate policy holding fixed current income and expectations about future income and inflation. Using the goods market-clearing condition the total effect is calculated as

$$x_t=E_t\sum_{T=t}^{\infty}\left(m\beta\right)^{T-t}\left[\left(1-\beta\right)x_T-\beta\sigma\pi_{T+1}\right]-\beta\sigma E_t\sum_{T=t}^{\infty}\left(m\beta\right)^{T-t}\left(R_T-\bar{r}\right)$$

using E_t for average beliefs.

9.1 Some basics

With probability $1-\nu$ interest rates remain at zero and with probability ν interest rates revert to steady state with $R_H=\bar{r}$. The high state under rational expectations satisfies $x_H=0$. In general, conditional expectations of output satisfy

$$E_t x_{T+1}=\left(1-\nu\right)^{T+1-t}x_L+\rho^{T-t}\omega_{t-1}^x$$

where the first two terms are the rational expectations component of forecasts and the third term the learning component. The assumption that prices are fixed implies $E_t\pi_{T+1}=0$ for all $T>t$. Suppose after $\bar{T}$ the economy returns to the high state with probability one. Then

$$\begin{aligned}E_t\sum_{T=t}^{\bar{T}}\left(m\beta\right)^{T-t}x_{T+1} &= \sum_{T=t}^{\bar{T}}\left(m\beta\right)^{T-t}\left(1-\nu\right)^{T+1-t}x_L+\sum_{T=t}^{\bar{T}}\left(m\beta\right)^{T-t}\rho^{T-t}\omega_{t-1}^x \\ &= \left(1-\nu\right)\frac{1-\left[m\beta\left(1-\nu\right)\right]^{\bar{T}+1}}{1-m\beta\left(1-\nu\right)}x_L+\frac{1-\left(m\beta\rho\right)^{\bar{T}+1}}{1-m\beta\rho}\omega_{t-1}^x.\end{aligned}$$

As $\bar{T}\to\infty$ we have

$$E_t\sum_{T=t}^{\infty}\left(m\beta\right)^{T-t}x_{T+1}=\frac{\left(1-\nu\right)}{1-m\beta\left(1-\nu\right)}x_L+\frac{1}{1-m\beta\rho}\omega_{t-1}^x.$$

For the interest rate we have

$$E_t R_{T+1}=\left(1-\left(1-\nu\right)^{T+1-t}\right)\bar{r}$$

since inflation is always zero. This is the interest rate forecast under rational expectations and learning given maintained assumptions. We then have

$$\begin{aligned} E_t \sum_{T=t}^{\bar{T}} (m\beta)^{T-t} R_{T+1} &= \sum_{T=t}^{\bar{T}} (m\beta)^{T-t} \left(1-(1-\nu)^{T+1-t}\right) \bar{r} \\ &= \left[\frac{1-(m\beta)^{\bar{T}-1}}{1-m\beta} - (1-\nu)\frac{1-[m\beta(1-\nu)]^{\bar{T}+1}}{1-m\beta(1-\nu)}\right] \bar{r}. \end{aligned}$$

In the limit $\bar{T} \to \infty$ we have

$$E_t \sum_{T=t}^{\infty} (m\beta)^{T-t} R_{T+1} = \left[\frac{1}{1-m\beta} - \frac{(1-\nu)}{1-m\beta(1-\nu)}\right] \bar{r}.$$

9.2 Rational Expectations: Proposition 3

Condition on being in the 'low' state with zero interest rates, we can evaluate expectations in the optimal consumption function to give

$$\begin{aligned} c_t(i) &= (1-\beta)x_t - \beta\sigma R_t + E_t^i \sum_{T=t}^{\infty} (m\beta)^{T-t} \left[(1-\beta)\beta m x_{T+1} - \beta\sigma(\beta m R_{T+1} - \bar{r})\right] \\ &= \left[(1-\beta) + (1-\beta)\beta m\left(\frac{(1-\nu)}{1-m\beta(1-\nu)}\right)\right] x_L + \beta\sigma\left(\frac{1}{m\beta\nu - m\beta + 1}\right)\bar{r} \\ &= \left[\frac{1-\beta}{1-m\beta(1-\nu)}\right] x_L + \beta\sigma\left(\frac{1}{1-m\beta(1-\nu)}\right)\bar{r} \end{aligned}$$

where again the second term is the pure partial equilibrium effect of the policy announcement.

Solving for aggregate output using

$$x_L = x_t = \int c_t(i)\, di$$

gives

$$x_L\left(\beta\frac{1-m(1-\nu)}{1-m\beta(1-\nu)}\right) = \beta\sigma\left(\frac{1}{1-m\beta(1-\nu)}\right)\bar{r}$$

or

$$x_L = \frac{\sigma}{1-m(1-\nu)}\bar{r}.$$

The output effect must then satisfy the decomposition

$$\begin{aligned} x^{PE} &= \left(\frac{\beta\sigma}{1-m\beta\left(1-\nu\right)}\right)\bar{r} \\ x^{GE} &= \frac{\sigma\left(1-\beta\right)}{\left(1-m\left(1-\nu\right)\right)\left(1-m\beta\left(1-\nu\right)\right)}\bar{r} \end{aligned}$$

In the limit as $m \to 1$ we have the benchmark new Keynesian model rational expectations solution

$$\begin{aligned} x^{PE} &= \left(\frac{\beta\sigma}{1-\beta\left(1-\nu\right)}\right)\bar{r} \\ x^{GE} &= \frac{\left(1-\beta\right)\sigma}{\nu\left(1-\beta\left(1-\nu\right)\right)}\bar{r}. \end{aligned}$$

As $\nu \to 0$ the general equilibrium component explodes.

9.3 Learning: Propositions 4 and 5

Under learning dynamics the model is given by

$$c_t\left(i\right) = \left(1-\beta\right)x_t - \beta\sigma R_t - \beta\sigma E_t^i \sum_{T=t}^{\infty}\left(m\beta\right)^{T-t}\left(\beta m R_{T+1} - \bar{r}\right) \tag{23}$$

$$\omega_{t+1|t}^{x} = \left(\rho - g\right)\omega_{t|t-1}^{x} + g x_t \tag{24}$$

where the second equation gives the Kalman updating of output beliefs. At the time of the policy announcement beliefs are consistent with the high state: $\omega_{0|-1}^{x} = 0$. We have the follow sequence of optimal consumption plans. In period 0 at the time of announcement:

$$c_0\left(i\right) = \left(1-\beta\right)x_0 + \frac{\beta\sigma}{1-m\beta\left(1-\nu\right)}\bar{r}$$

where the second term is the partial equilibrium effect of forward guidance. The total effect then follows from market clearing

$$x_0 = \frac{\sigma}{1-m\beta\left(1-\nu\right)}\bar{r}$$

providing the decomposition

$$\begin{aligned} x_0^{GE} &= x_0 - x_0^{PE} \\ &= \frac{\sigma}{1 - m\beta(1-\nu)}\bar{r} - \frac{\beta\sigma}{1 - m\beta(1-\nu)}\bar{r} \\ &= \frac{(1-\beta)\sigma}{1 - m\beta(1-\nu)}\bar{r}. \end{aligned}$$

In period 1 we have

$$\begin{aligned} c_1(i) &= (1-\beta)x_1 + \frac{m\beta(1-\beta)}{1-m\beta\rho}\omega_{1|0}^x + \frac{\beta\sigma}{1 - m\beta(1-\nu)}\bar{r} \\ &= (1-\beta)x_1 + \frac{m\beta(1-\beta)}{1-m\beta\rho}gx_0 + \frac{\beta\sigma}{1 - m\beta(1-\nu)}\bar{r} \end{aligned}$$

where the second equality follows from the updating of beliefs. The partial equilibrium effect in period 1 is again given by the final term and the total effect of policy

$$x_1 = \frac{m(1-\beta)}{1-m\beta\rho}gx_0 + \frac{\sigma}{1 - m\beta(1-\nu)}\bar{r}$$

with general equilibrium effect given by

$$\begin{aligned} x_1^{GE} &= x_1 - x_1^{PE} \\ &= \frac{m(1-\beta)}{1-m\beta\rho}gx_0 + \frac{(1-\beta)\sigma}{1 - m\beta(1-\nu)}\bar{r}. \end{aligned}$$

The general equilibrium effect in period 1 therefore differs to that in period 0.

Repeating these calculations progressively and recalling $g = \gamma\rho$ establishes the follow expressions. The total effect on output in any future period $T > 0$ is

$$x_T = \frac{m(1-\beta)}{1-m\beta\rho}\gamma\rho\sum_{j=0}^{T-1}\left[\rho(1-\gamma)\right]^{T-j-1}x_j + \frac{\sigma}{1 - m\beta(1-\nu)}\bar{r}$$

with partial and general equilibrium effects

$$\begin{aligned} x_T^{PE} &= \frac{\beta\sigma}{1 - m\beta(1-\nu)}\bar{r} \\ x_T^{GE} &= \frac{m(1-\beta)}{1-m\beta\rho}\gamma\rho\sum_{j=0}^{T-1}\left[\rho(1-\gamma)\right]^{T-j-1}x_j + \frac{(1-\beta)\sigma}{1 - m\beta(1-\nu)}\bar{r}. \end{aligned}$$

We can write the general equilibrium effect as a first order difference equation by noting

$$
\begin{aligned}
x_{T+1}^{GE} &= \frac{m(1-\beta)}{1-m\beta\rho}\gamma\rho\sum_{j=0}^{T}\left[\rho(1-\gamma)\right]^{T-j-1}x_j+\frac{(1-\beta)\sigma}{1-m\beta(1-\nu)}\bar{r} \\
&= \frac{m(1-\beta)}{1-m\beta\rho}\gamma\rho x_T^{GE}+\rho(1-\gamma)\frac{m(1-\beta)}{1-m\beta\rho}\gamma\rho\sum_{j=0}^{T-1}\left[\rho(1-\gamma)\right]^{T-j-1}x_j+\frac{(1-\beta)\sigma}{1-m\beta(1-\nu)}\bar{r} \\
&= \frac{m(1-\beta)}{1-m\beta\rho}\gamma\rho x_T^{GE}+\rho(1-\gamma)\left[x_T^{GE}-\frac{(1-\beta)\sigma}{1-m\beta(1-\nu)}\bar{r}\right]+\frac{(1-\beta)\sigma}{1-m\beta(1-\nu)}\bar{r} \\
&= \left[\frac{m(1-\beta)}{1-m\beta\rho}\gamma\rho+\rho(1-\gamma)\right]x_T^{GE}+\frac{\left[1-\rho(1-\gamma)\right](1-\beta)\sigma}{1-m\beta(1-\nu)}\bar{r} \\
&= \Psi_x x_T^{GE}+\Psi_r\bar{r}
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi_x &= \frac{m(1-\beta)}{1-m\beta\rho}\gamma\rho+\rho(1-\gamma) \\
\Psi_r &= \frac{\left[1-\rho(1-\gamma)\right](1-\beta)\sigma}{1-m\beta(1-\nu)}.
\end{aligned}
$$

This establishes proposition 4.

For corollary 2 note that this process has a mean given by

$$
\frac{\Psi_r}{1-\Psi_x}\bar{r}.
$$

Standard calculations then give the various results that are stated.

Finally, for proposition 5, note that the mean can be written as

$$
\begin{aligned}
\frac{\Psi_r}{1-\Psi_x} &= \frac{1}{1-\left[\frac{m(1-\beta)}{1-m\beta\rho}\gamma\rho+\rho(1-\gamma)\right]}\frac{\left[1-\rho(1-\gamma)\right](1-\beta)\sigma}{1-m\beta(1-\nu)} \\
&= \left(\frac{(1-\beta)\sigma}{\nu(1-\beta(1-\nu))}\right)\left(\frac{\left[1-\rho(1-\gamma)\right]}{1-\left[\frac{m(1-\beta)}{1-m\beta\rho}\gamma\rho+\rho(1-\gamma)\right]}\frac{\nu(1-\beta(1-\nu))}{1-m\beta(1-\nu)}\right) \\
&= \left(\frac{(1-\beta)\sigma}{\nu(1-\beta(1-\nu))}\right)\left(\frac{1-\rho(1-\gamma)}{1-\rho(1-\mu\gamma)}\times\frac{\nu(1-\beta(1-\nu))}{1-m\beta(1-\nu)}\right)
\end{aligned}
$$

where

$$
\mu = 1-\frac{m(1-\beta)}{1-m\beta\rho}
$$

and the first term in parenthesis is the rational expectations general equilibrium effect. When

$m = 1$ we have

$$\frac{\Psi_r}{1-\Psi_x} = \left(\frac{(1-\beta)\,\sigma}{\nu\,(1-\beta\,(1-\nu))}\right)\left(\frac{1-\rho\,(1-\gamma)}{1-\rho\,(1-\mu\gamma)} \times \nu\right).$$

The first term in parenthesis is the general equilibrium effect under rational expectations. The general equilibrium effect under learning is therefore larger than rational expectations when

$$\frac{1-\rho\,(1-\gamma)}{1-\rho\,(1-\mu\gamma)} \times \nu > 1$$

or

$$\left(\frac{1-\beta\rho}{1-\rho}\right)\frac{1-\rho\,(1-\nu)}{1-\beta\rho\,(1-\gamma)} \times \nu > 1.$$

For ρ and γ sufficiently large this inequality will be satisfied for given ν. For example, as $\rho \to 1$ the inequality is always satisfied because the general equilibrium effect is a unit root process.

10 Robustness: Further Results

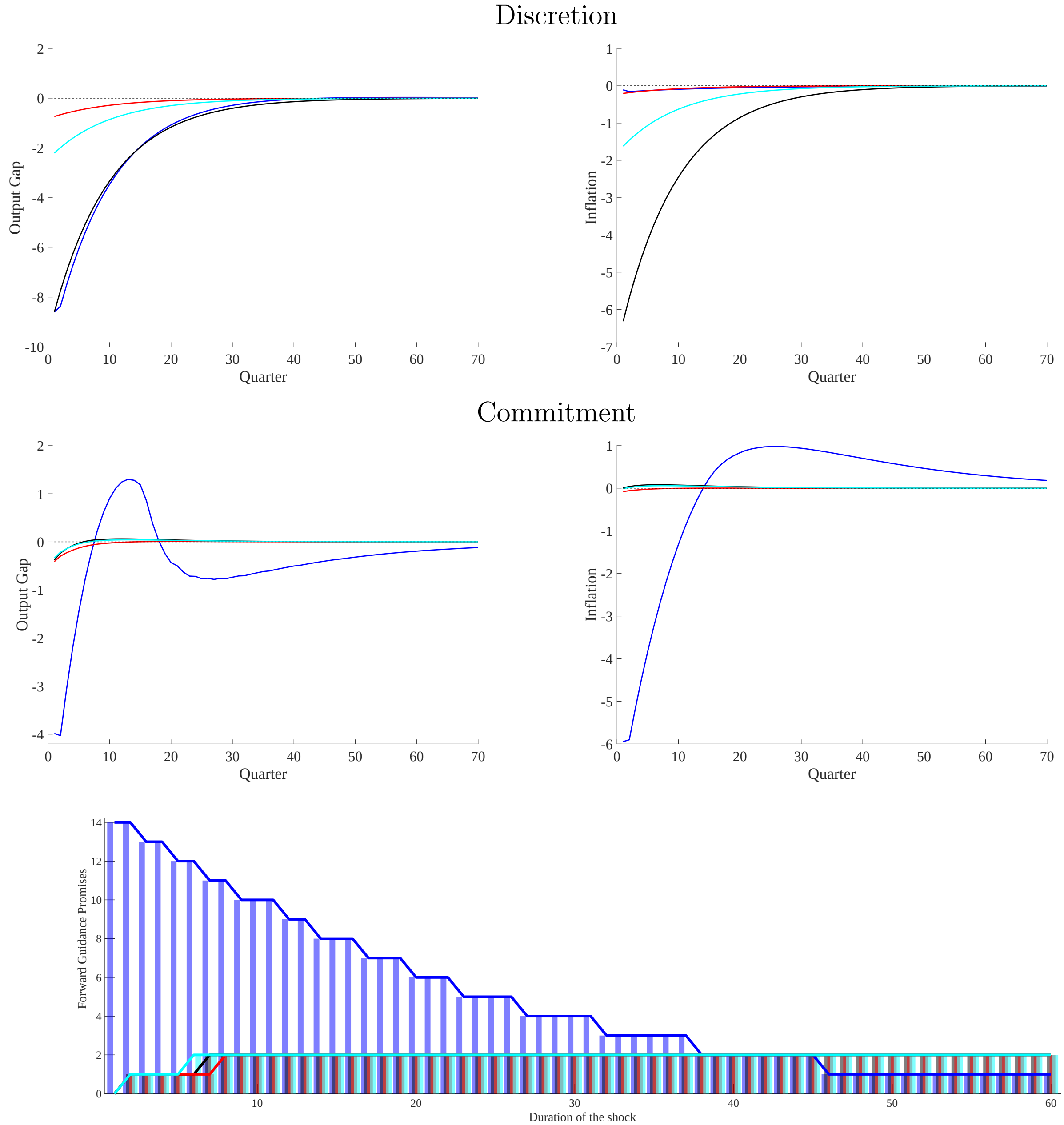
10.1 Further exploration of optimal policy in alternative models

To compare optimal policy under the different frameworks in Section 6.1 in the main draft, we select different sized shocks for each model so that the same sized output gap is predicted on impact under discretion. In this section, we investigate how the conclusions change if we assume different normalization. We consider three alternative in this sections:

1. We fix the size of the shock, r_L, to be the same in each model.
2. We fix the size of the shock, r_L, to be the same in each model and we remove the normalizing constant in the learning model.
3. We remove the normalizing constant in the learning model and choose a shock so that the same impact on the output gap is observed in each model under discretion.

Figure 9 shows optimal policy under the first case. By assumption, the learning model and RE predict the same impact of the shock under discretion. The impacts of the shock under the other two assumptions though are almost an order of magnitude smaller. This reflects the additional assumptions in each of the models that weigh against the general equilibrium effects of the shocks. Because of the smaller impact of the shocks in the latter two cases, the amount of forward guidance necessary to stabilize expectations is significantly reduced. Although, the profile of the state contingent promises remains the same.

Figure 9: Policy in alternative models - I



Notes: The same natural rate shock is imposed for the standard model (black), learning model (blue), BNK model (red), and a THANK model (cyan). Shared parameter values across all models are $\beta = 0.99$, $\sigma = 0.5$, $\delta = 0.1$, $\kappa = 0.02$ and $\lambda_x = 0.05$; RE shock parameter $r_L = -0.003$; learning model parameters $g = 0.075$, $\rho = 0.985$, $r_L = -0.003$; BNK parameters: $M = 0.85$ and $M^f = 0.8$. THANK parameters: $M = 0.9701$, $M^f = 1$, and $N = 0.843$.

Figure 10 show optimal policy with the normalizing constant removed from the learning model. The agents in this case are assumed to only anticipated the direct effect of the shock and do not contemplate any general equilibrium effects except those that operate through the learning assumptions. Without the normalization the impact of the shock is comparable to that predicted by the Behavioral New Keynesian model. A result that was foreshadowed by the simple analytics of forward guidance discussed in Section 5 of the main text. Because the shock has a much smaller impact, the size of the forward guidance promises necessary to stabilize the economy are greatly reduced. The profile of the state contingent promises though remains unchanged, i.e., policy remains front loaded and the insurance principle still applies.

Figure 11 shows optimal policy with the normalizing constant removed from the learning model and the shock scaled so that output gaps is the same as RE discretion. We consider this case because during the pandemic inflation was much more stable relative to the movements in the output gap. Therefore, the predictions of the model under RE are hard to reconcile with this time period. By assumption, the same impact on inflation is inherited in the learning model. This case shows that this counterfactual prediction is due solely to our normalization choice. Removing it reveals that our model predicts very small movements in inflation when there is a significant output gap similar to the Behavioral New Keynesian model. The profile of optimal forward guidance policy, however, remains qualitatively the same.

That the character of optimal forward guidance policy is robust to these different assumptions in the learning model reflects the fact that the general equilibrium effects of forward guidance policy are always back-loaded. Learning is the critical assumption.

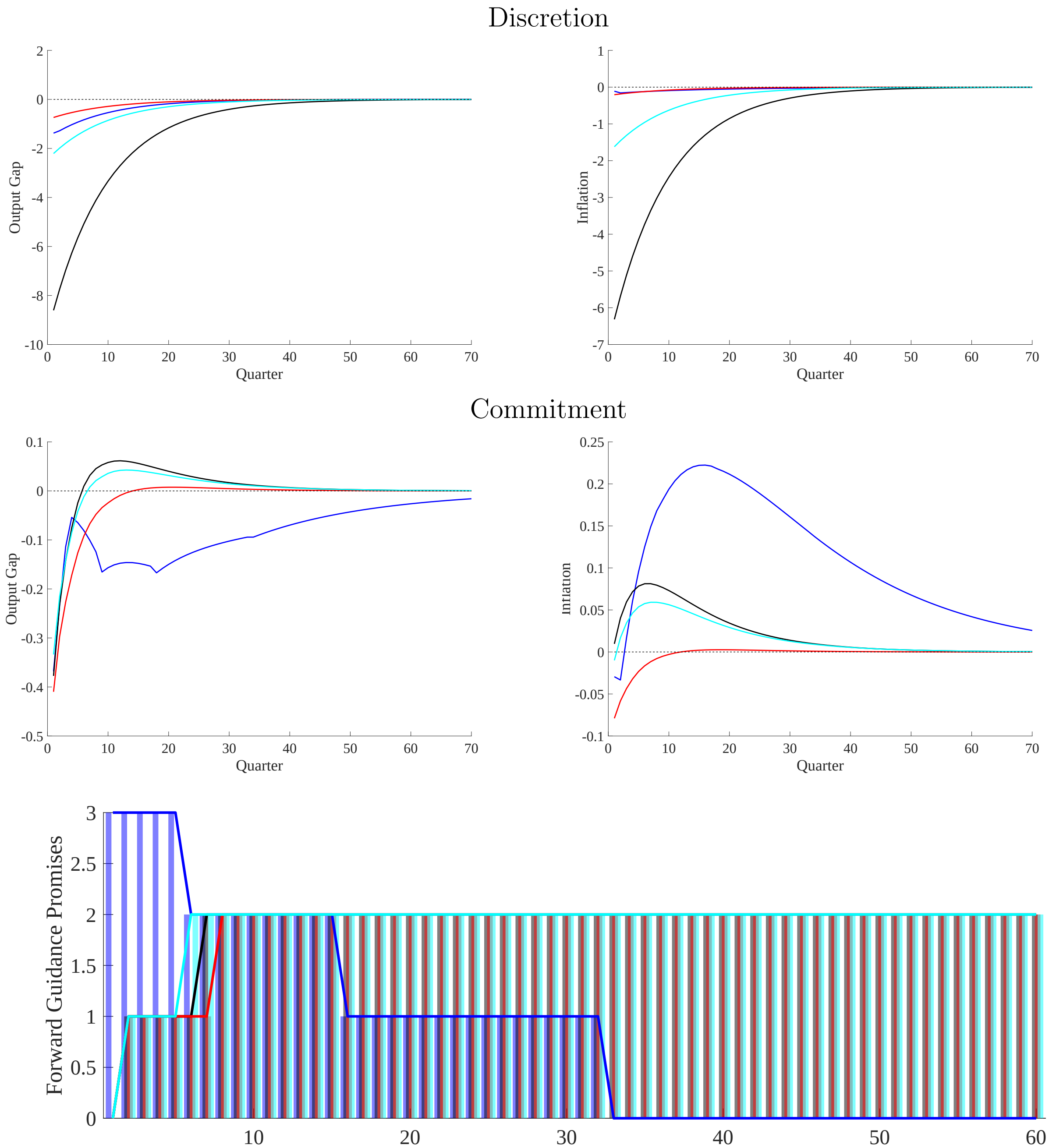
10.2 Markdowns and markups in initial expectations

To show how learning affects the power of forward guidance policy, Figure 12 shows how optimal forward guidance policy changes if inflation expectations are not at steady state when the shock occurs. Instead, we compute optimal policy for the case where inflation expectations are $\pm$ 0.5% above and below steady state, respectively, when the shock occurs. Figure 12 shows optimal policy for our baseline calibration of the learning parameters. More forward guidance is needed to offset an additional fall inflation expectations and less is needed if inflation expectations are higher. The overall profile of policy in unchanged.

10.3 Central bank preferences

The final exploration that we consider is the effect that central bank preferences over inflation stabilization relative to output stabilization have on optimal forward guidance policy. The parameter of interest is λ_x. Figure 13 shows the optimal policy predictions for four different

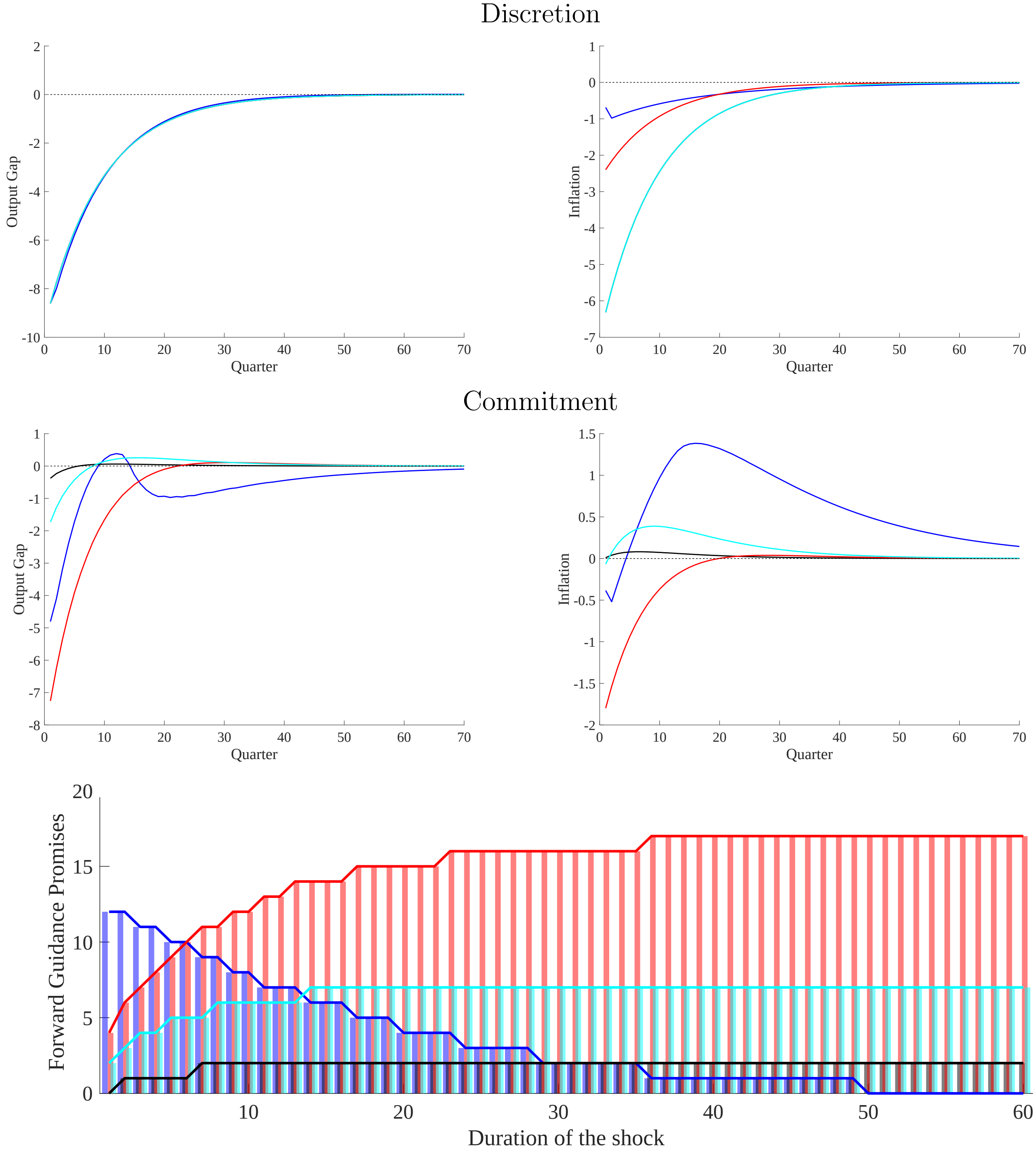
Figure 10: Policy in alternative models - II



Notes: The learning model no longer assumes any knowledge of the RE discretion solution. The same natural rate shock is imposed for the standard model (black), learning model (blue), BNK model (red), and a THANK model (cyan). Shared parameter values across all models are $\beta = 0.99$, $\sigma = 0.5$, $\delta = 0.1$, $\kappa = 0.02$ and $\lambda_x = 0.05$; RE shock parameter $r_L = -0.003$; learning model parameters $g = 0.075$, $\rho = 0.985$, $r_L = -0.003$; BNK parameters: $M = 0.85$ and $M^f = 0.8$. THANK parameters: $M = 0.9701$, $M^f = 1$, and $N = 0.843$.

settings of λ_x. The black lines corresponds to exact calibration considerd in Eggertsson and Woodford (2003). The red lines correspond to our baseline calibration in the paper. The

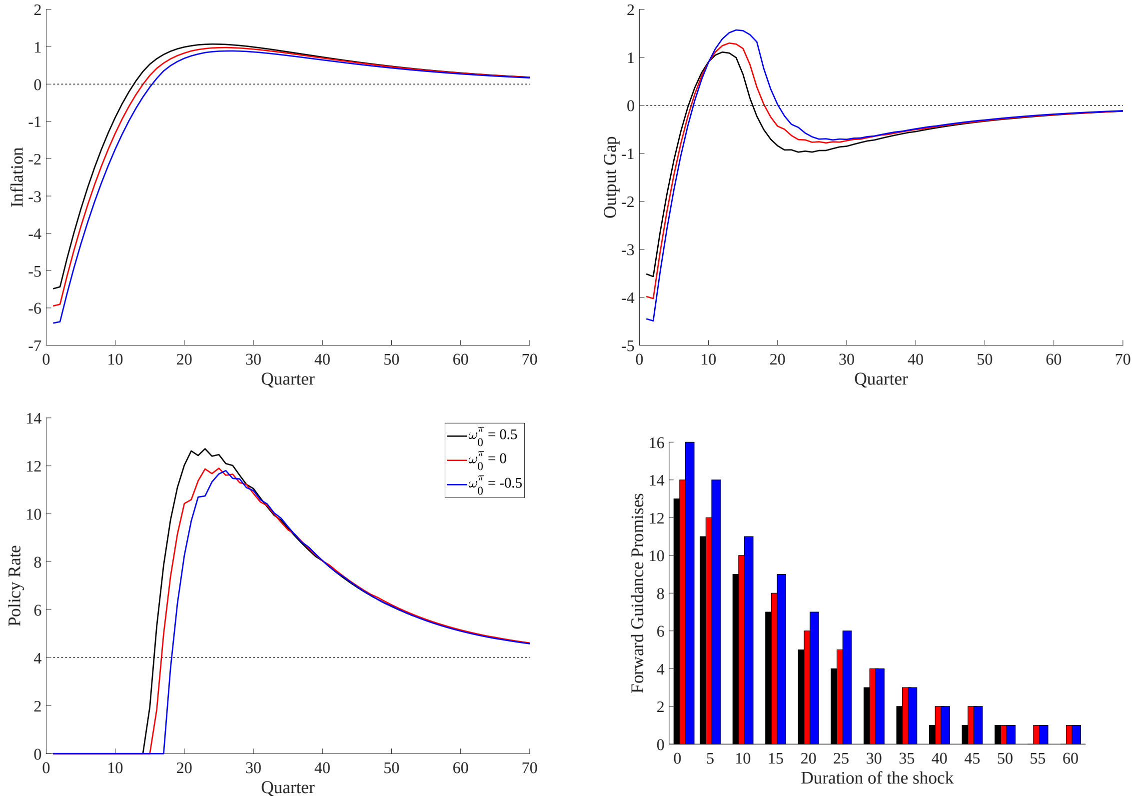
Figure 11: Policy in alternative models - III



Notes: The learning model no longer assumes any knowledge of the RE discretion solution. The natural rate shock is normalized so that it has the same impact in period one on the output gap under discretion for the standard model (black), learning model (blue), BNK model (red), and a THANK model (cyan). Shared parameter values across all models are $\beta = 0.99$, $\sigma = 0.5$, $\delta = 0.1$, $\kappa = 0.02$ and $\lambda_x = 0.05$; RE shock parameter $r_L = -0.003$; learning model parameters $g = 0.075$, $\rho = 0.985$, $r_L = -0.0187$; BNK parameters: $M = 0.85$, $M^f = 0.8$, and $r_L = -0.035$. THANK parameters: $M = 0.9701$, $M^f = 1$, $N = 0.843$, and $r_L = -0.012$.

blue lines show the assumption used in Section 6.3. The impact of this parameter has very

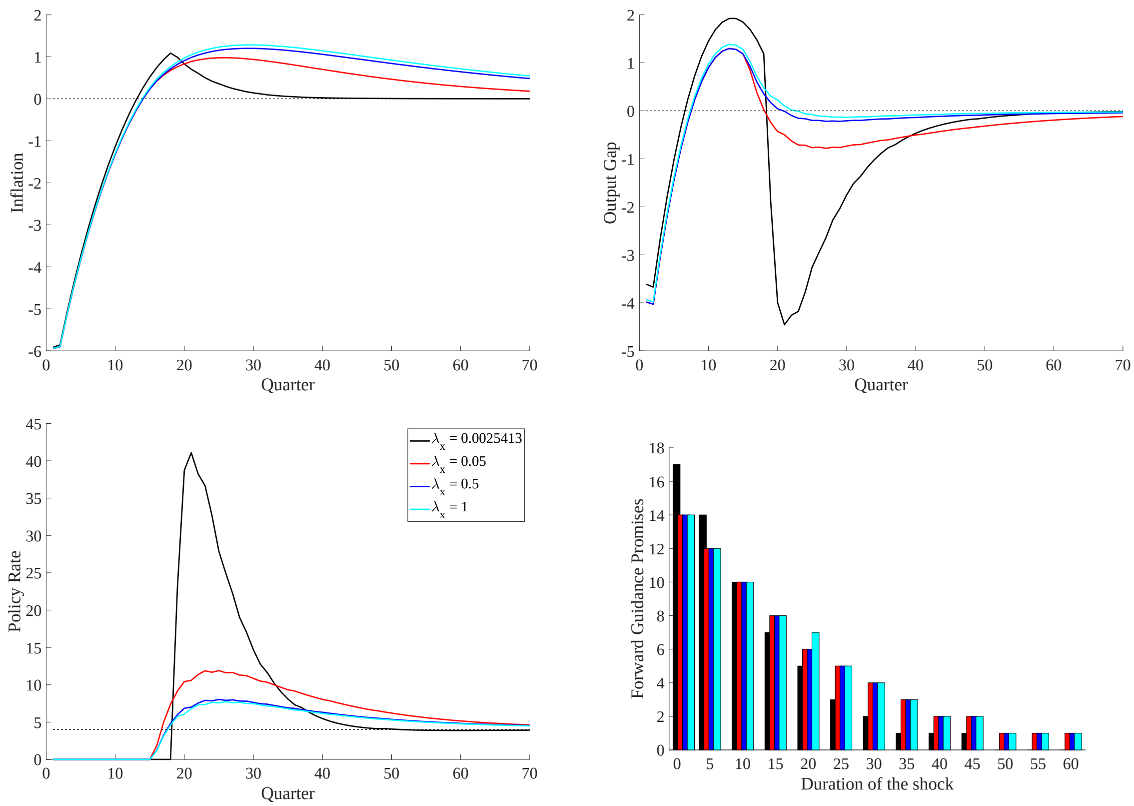
Figure 12: Markdowns and markups in initial expectations



Notes: Parameter values $g = 0.075$, $\rho = 0.985$, $\beta = 0.99$, $\sigma = 0.5$, $\delta = 0.1$, $\kappa = 0.02$, $\lambda_x = 0.05$, $r_L = -0.003$.

little effect on the profile of promised forward guidance. Its main impact is on managing the economy once the ZLB no longer binds.

Figure 13: Unanchored Expectations: Optimal Policy and the Central Bank's Loss Function



Notes: Parameter values $g = 0.075$, $\rho = 0.985$, $\beta = 0.99$, $\sigma = 0.5$, $\delta = 0.1$, $\kappa = 0.02$, $r_L = -0.003$.

References

Angeletos, G.-M., and C. Lian (2018): "Forward Guidance without Common Knowledge," *American Economic Review*, 108(9), 2477–2512.

Bilbiie, F. O. (2018): "Monetary Policy and Heterogeneity: An Analytical Framework," CEPR Discussion Papers 12601, C.E.P.R. Discussion Papers.

Del Negro, M., M. Giannoni, and C. Patterson (2012): "The forward guidance puzzle," Staff Reports 574, Federal Reserve Bank of New York.

Eggertsson, G., and M. Woodford (2003): "The Zero Bound on Interest Rates and Optimal Monetary Policy," *Brookings Papers on Economic Activity*, pp. 139–211.

Eggertsson, G. B., S. K. Egiev, A. Lin, J. Platzer, and L. Riva (2021): "A Toolkit for Solving Models with a Lower Bound on Interest Rates of Stochastic Duration," *Review of Economic Dynamics*.

Farhi, E., and I. Werning (2019): "Monetary Policy, Bounded Rationality, and Incomplete Markets," *American Economic Review*, 109(11), 3887–3928.

Gabaix, X. (2020): "A Behavioral New Keynesian Model," *American Economic Review*, 110(8), 2271–2327.

Gali, J. (2008): *Monetary Policy, Inflation, and the Business Cycle: An Introduction of the New Keynesian Framework*. Princeton University Press.

McKay, A., E. Nakamura, and J. Steinsson (2017): "The Discounted Euler Equation: A Note," *Economica*, 84(336), 820–831.

Sargent, T. J., and N. Williams (2005): "Impacts of Priors on Convergence and Escape Dynamics," *Review of Economic Dynamics*, 8(2), 360–391.

Woodford, M. (2003): *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.