Does my model predict a forward guidance puzzle?*

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Abstract

We show how to characterize the economic forces that generate the forward guidance puzzle in a wide variety of structural macroeconomic models. We accomplish this by showing that studying the predictions of forward guidance announcements is essentially the same as conducting E-stability analysis under adaptive learning. We show that the Iterative E-stability criterion identifies all of the most prominent forward guidance puzzle resolutions proposed in the literature, provides ways to evaluate their robustness, shows how new resolutions may be constructed, and is scalable to quantitatively relevant models. We show some common resolutions are robust while others are not. We also devise a novel solution to the forward guidance puzzle: sunspots.

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1 Introduction

A near ubiquitous feature of standard rational expectations (RE) structural monetary policy models is that credible promises to hold interest rates at zero for extended periods of time can generate significant jumps in output and inflation in the period the policy is announced. Moreover, the contemporaneous impact of a fixed future policy intervention can be made arbitrarily large today, when interest rates are constrained, simply by pushing the actual implementation of the policy farther into the future, a phenomenon known as the *forward guidance puzzle*. Since this feature of structural monetary models was first pointed out by papers such as Del Negro, Giannoni and Patterson (2012) and Carlstrom, Fuerst and Paustian (2015), a number of authors have sought to ameliorate and explain away this puzzle using, for example, credibility (Haberis, Harrison and Waldron, 2019), imperfect information (Carlstrom et al., 2015; Kiley, 2016), bounded rationality or level-k thinking (Gabaix, 2020; Angeletos and Lian, 2018; Farhi and Werning, 2019), life-cycle considerations (Del Negro et al., 2012; Eggertsson and Mehrotra, 2014; Eggertsson, Mehrotra and Robbins, 2019), heterogeneous agents with incomplete markets (McKay, Nakamura and Steinsson, 2016; Bilbiie, 2020), or the fiscal theory of the price level (Cochrane, 2017; McClung, 2021) to name just a few.

The source of the puzzle is the backward induction that agents (and the modeler) do to find the path that endogenous variables must take to arrive at the announced future state of the economy. Starting with the beliefs that must prevail when the announced policy terminates, one walks expectations iteratively backwards in time through the structural equations. This backward iteration implies an unstable dynamic in most New Keynesian models when monetary policy is constrained by the zero lower bound (ZLB) on nominal interest rates. Inflation and output beliefs diverge toward positive or negative infinity with each deduction, which generates larger and larger jumps in these variables today, the farther the terminal date of the announced policy. In other words, there is no smooth path from where agents believe the economy will be in the future to where the economy is today in many modeled environments.

The search for a smooth path from some arbitrary set of beliefs to a specific rational expectation solution is not a new problem. The expectations literature, typified by Adaptive Learning as in Evans and Honkapohja (2001), has studied this problem for decades. The problem is just framed with time running in the opposite direction. Instead of asking how we go from some belief about the future back to today, this literature asks how, starting from some arbitrary beliefs, we can arrive over time to a specific belief. We show that assessing the power of forward guidance and studying the Expectational stability (E-stability) properties of a Rational Expectations Equilibrium (REE) are essentially one and the same. Specifically, the notion of Iterative Expectational stability (IE-stability) first proposed by Evans (1985) provides sufficient conditions that allow one to determine if any model, or more precisely, any equilibrium of a model is likely to predict puzzling behavior in response to a forward guidance announcement.[1] The conditions allow for a direct diagnosis of the economic assumptions that are driving the explosive beliefs across a wide variety of models and can be used for equilibrium selection.

We show that viewing the forward guidance puzzle through the lens of IE-stability provides a clear economic interpretation for why the puzzle exists. IE-stability is a local stability condition for the equilibrium dynamics of expectations. Stability depends on both the features of the economy at the ZLB and the economy after liftoff. An REE fails to be IE-stable when, for a given initial belief, endogenous variables of the economy respond more than one-for-one to expectations. For example, if higher inflation expectations trigger a rise in inflation over and above that implied by the expectation alone, then inflation and inflation expectations will continue to diverge when agents are learning. For a forward guidance announcement this dynamic happens in reverse. Agents contemplate some future belief about inflation in some period T^* and ask what realized inflation rate in the previous period, T^*-1, rationalizes that belief. If it is an *even* higher inflation rate, then that implies inflation expectations in period T^*-2 should be higher and hence realized inflation in T^*-2 is higher again. Repeating the deduction over and over again leads to the unbounded response of inflation to the policy. This is exactly what occurs in the New Keynesian model when monetary policy is constrained by the ZLB. Higher inflation expectations lead directly to a lower real interest rate, which leads to higher consumption, and then further increases in inflation and inflation expectations. This dynamic repeats with every additional anticipated period of zero interest rates. Dampening or breaking this feedback loop is what all forward guidance puzzle resolutions must do. By connecting

[1] IE-stability predates the more well-known concept of E-stability. Evans (1983), Evans (1985), and Evans (1986) all propose iterative procedures as an equilibrium selection criterion, which Evans (1985) defined as "Expectational Stability." Later, Evans (1989) showed that this criterion shared features with the convergence conditions for agents who engage in least squares learning studied in Marcet and Sargent (1989). Further contrasts between the two approaches are discussed in Evans and Honkapohja (1992) and the current terminology of E-stability and IE-stability dates back to Evans and Honkapohja (1994).

IE-stability and forward guidance, we identify three (not mutually exclusive) categories of resolution strategies that enhance or change the stability properties of models studied under adaptive learning and which mitigate or solve the puzzle:

1. *Over-discounting expectations:* Adding assumptions that imply increased discounting of the future relative to the standard representative agent model. Examples include myopia or life-cycle dynamics where reduced planning horizons naturally decrease the weight given to future events.

2. *Predetermining expectations:* Adding assumptions that introduce new state variables to the standard representative agent model. Examples include sticky information models where beliefs are explicit state variables or sunspot equilibria where some subset of expectations depend in part on some non-fundamental state.

3. *History dependent policy:* Assuming monetary or fiscal policy rules that explicitly depend on the endogenous outcomes observed during the ZLB period. Examples include price level targeting (with some caveats) and active fiscal policy regimes. Expectations of retroactive policy adjustments *after* liftoff offset the impacts of forward guidance policy.

The first category of resolutions dampen the unstable dynamics in the model without altering its reduced form such as in the Behavioral New Keynesian model of Gabaix (2020). Categories two and three change the dynamics of expectations by introducing new state variables. The third category follows from the fact that the forward guidance puzzle is always a two-regime phenomenon. Studying a model with an interest rate peg is not prescriptive for whether a forward guidance puzzle will occur without specifying the policy pursued after the peg is abandoned.

This categorization and connection to IE-stability provides a road map for where to look for resolutions of the puzzle by appealing to the large body of knowledge of model features that enhance expectational stability. It also distinguishes between resolutions to the forward guidance puzzle that scale with the complexity of the model, i.e., increasing the number of state variables, and indicates a proposed resolution's potential robustness. For example, because the first category of resolutions seek only to dampen the existing general equilibrium effects, these types of resolutions are sensitive to parameter choices. In particular, we show that increasing the flexibility of prices by very small amounts in the basic Behavioral New Keynesian model may undue the dampening implied by the over-discounting of expectations and bring back the forward guidance puzzle. Consequently, solutions which rely on over-discounting in simple models may not scale to larger and more quantitatively relevant applications. The latter two resolution categories, though, are more robust. They seek to fundamentally change the dynamics of expectations and their general equilibrium effects.

Connecting the forward guidance puzzle to IE-stability also sheds light on two additional aspects of modeling zero interest rates and forward guidance announcements debated in the literature. First, there is nothing special about monetary policy forward guidance and the ZLB that generates this dynamic. Any equilibrium of a model that fails to meet our IE-stability criteria may result in puzzling behavior for a forward guidance announcement of any kind. For example, our IE-stability conditions are predictive for the forward fiscal guidance puzzle explored by Canzoneri, Cao, Cumby, Diba and Luo (2018) or for when large positive/negative impacts of announced disinflation arise such as in Ball (1994) and Gibbs and Kulish (2017). Second, IE-stability and our framework show that the forward guidance puzzle is not a by-product of indeterminacy as suggested by Carlstrom et al. (2015). It is well-known that E-stability and determinacy are distinct concepts.[2] In fact, in the class of models we consider the existence of multiple stationary equilibria implies the existence of sunspot solutions that satisfy the IE-stability condition and therefore do not exhibit the forward guidance puzzle. In these puzzle-free equilibria, extraneous sunspots predetermine expectations in a manner that counteracts the effects of the policy announcements.[3]

We are not the first to point out that indeterminacy can be exploited to resolve the forward guidance puzzle. Cochrane (2017) shows the existence of multiple equilibria at the ZLB in a standard New Keynesian environment of which many do not exhibit the features of the forward guidance puzzle. What we provide in addition to this insight is a framework for equilibrium selection that scales to a broad class of models. Our method can isolate the individual economic assumptions that contribute to the puzzle allowing for puzzle resolutions that are derived from

[2]The connection/disconnection between determinacy and E-stability has been widely studied, for example, in McCallum (2007) and Bullard and Eusepi (2014). Ellison and Pearlman (2011) also extends this analysis to IE-stability.

[3]Note that indeterminate models generically admit both minimal state variable (MSV) solutions and sunspot solutions that depend on extraneous sunspot shocks. This paper considers both solution types; section 3 considers the MSV solution(s) of determinate and indeterminate models, whereas section 5 constructs puzzle-free sunspot equilibria.

the micro-foundations of the model rather than by selecting for the macro predictions one finds most palatable as in Cochrane's framework.

We provide a simple example of the connection between the forward guidance puzzle and IE-stability in the next section. Section 3 defines the forward guidance puzzle in a general way and shows how IE-stability can characterize its existence. Section 4 applies IE-stability to a general New Keynesian model that nests the models of Gabaix (2020) and Bilbiie (2020) to explore puzzle resolutions that rely on over-discounting of expectations and the robustness of these modeling strategies to resolve the forward guidance puzzle.

We then explore two extensions. In section 5, we provide an extension of the indeterminate solution techniques of Bianchi and Nicolò (2021) to anticipated structural change. We show how these solutions may be analyzed using IE-stability in the same way as determinate solutions. We use these techniques to characterize when the forward guidance puzzle exists in the standard three equation New Keynesian model for both unique RE solutions and non-unique sunspot RE solutions. We show that IE-stability is a necessary and sufficient condition in the model for ruling out the forward guidance puzzle in the determinate case, and that for a class of sunspot solutions, IE-stability is always satisfied, which rules out the puzzle. Lastly, we show that the sunspot resolution of the forward guidance puzzle scales to the medium-scale model of Smets and Wouters (2007) estimated on data from the United States.

Section 6 extends the IE-stability analysis to models with Markov-switching policy and explores an application to history dependent monetary and fiscal policies. Cochrane (2017) and McClung (2021) show that expectations of active fiscal policy and passive monetary policy resolves the forward guidance puzzle. This policy introduces history dependence at the ZLB because a promise to fix the path of the nominal interest rate affects the real value of debt, which in turn influences inflation and inflation expectations beyond the forward guidance horizon. Expectations of policy choices after liftoff then restrain inflation when the economy is at the ZLB, eliminating the puzzle. We illustrate this in a standard modeling environment with exogenous Markov-switching between active and passive fiscal policy regimes upon liftoff from the ZLB. Importantly, IE-stability and determinacy of an REE at the ZLB in this model do not overlap for a wide range of the plausible parameterizations, which illustrates that IE-stability is the relevant criteria to study when assessing the power of forward guidance.

Finally, leveraging the Markov-switching machinery, we study forward guidance for a central bank that pursues price level targeting upon liftoff. We show that price level targeting on its own mitigates but does not eliminate all aspects of the forward guidance puzzle. We introduce an exogenous possibility that policymakers might renege on the promised forward guidance policy in this scenario. We show that expectations of reneging on policy, or alternatively imperfect central bank credibility, resolves the puzzle.

Related Literature. There are many other papers that solve or at least mitigate the forward guidance puzzle using one of the three resolutions mechanisms that IE-stability identifies. Additional examples of resolutions that rely on history dependent policy to resolve the puzzle are proposed by Bilbiie (2018) and Diba and Loisel (2021) who employ price level targeting and endogenous money growth rules, respectively. Additional examples of resolutions that rely on predetermining expectations are proposed by Kiley (2016) and Gorodnichenko and Sergeyev (2021) who use sticky information and an exogenous zero lower bound on inflation expectations, respectively; and Eggertsson and Mehrotra (2014) and Gibbs (2018) who each use a downward rigidity in nominal wages. Additional papers that rely on bounded rationality and information frictions are Angeletos and Sastry (2021), Eusepi, Gibbs and Preston (2022), and Evans, Gibbs and McGough (2022). The former paper relies on either imperfect information or myopia/bounded rationality - Angeletos and Huo (2021) show an equivalence result for the two deviations from full information rational expectations - while the latter two papers rely on adaptive learning and forms of level-k reasoning.

The novel sunspot resolution to the forward guidance puzzle that we derive relies heavily on the solution methods of Cagliarini and Kulish (2013) and Kulish and Pagan (2017) combined with the methods of Bianchi and Nicolò (2021) for solving for sunspot solutions. We generalize the Bianchi and Nicolò (2021) method to capture the zero lower bound and show how to adapt it to the reduced form of Binder and Pesaran (1997).

2 Insights from a Simple Model

A RE solution to a forward guidance announcement and the study of the same model under adaptive learning are in practice identical exercises with one exception: time's arrow runs

Table 1: Expectations and Anticipation

	RE	AE
Unanticipated ($T^a = T^*$)	$\pi_t = \begin{cases} -\frac{1}{\phi}\bar{i} & \text{if } t = T^a \\ 0 & \text{if } t < T^a \end{cases}$	$\pi_t = \begin{cases} -\left(\frac{1}{\phi}\right)^{-T^a+t+1}\bar{i} & \text{if } T^a \leq t \\ 0 & \text{if } t < T^a \end{cases}$
Anticipated ($T^a < T^*$)	$\pi_t = \begin{cases} -\left(\frac{1}{\phi}\right)^{T^*-t+1}\bar{i} & \text{if } T^a \leq t \leq T^* \\ 0 & \text{if } t < T^a \end{cases}$	$\pi_t = \begin{cases} -\frac{1}{\phi}\bar{i} & \text{if } t = T^* \\ 0 & \text{if } t < T^* \end{cases}$

Notes: Solutions to one-time unanticipated and anticipated monetary policy shocks, $\bar{i} < 0$, under rational expectations (RE) and adaptive expectations (AE).

in the opposite direction in each case. To illustrate, compare the equilibrium outcomes of an unanticipated versus anticipated monetary policy shock when agents have either RE or adaptive expectations (AE), where $\hat{\mathbb{E}}_t\pi_{t+1} = \pi_{t-1}$.

Consider these shocks in a simple Fisher model

$$i_t = \hat{\mathbb{E}}_t\pi_{t+1}, \tag{1}$$

where i, π are deviations from steady state (i.e. $i = \pi = 0$ in steady state), and $\hat{\mathbb{E}}$ denotes either RE or AE. The central bank sets the nominal interest rate i according to a policy rule: $i_t = \bar{i} + \phi\pi_t$, where $\bar{i}$ is a deterministic shock that is zero unless stated otherwise. Inflation in this economy evolves according to

$$\pi_t = -\frac{1}{\phi}\bar{i} + \frac{1}{\phi}\hat{\mathbb{E}}_t\pi_{t+1}.$$

The usual forward guidance thought experiment is to compare the response of inflation to a one-time change in $\bar{i} < 0$ while varying the timing of its implementation, $t = T^*$, and timing of its announcement to the public, $t = T^a$. An unanticipated policy is one where $T^* = T^a$ and an anticipated or forward guidance policy is one where $T^a < T^*$. The response of inflation to a shock is different depending on how policy is specified and the expectations assumption.

Table 1 gives the solution paths of inflation to a one-time monetary policy shock, $\bar{i} < 0$, that is either unanticipated or anticipated under RE and AE over a period of interest. The AE solution is calculated by substituting in $\hat{\mathbb{E}}_t\pi_{t+1} = \pi_{t-1}$ and rolling the economy forward in time. The RE solution for the anticipated shock is calculated by backward induction. Starting in $t = T^*$, $\mathbb{E}_t\pi_{t+1} = 0$ is imposed and the economy is rolled backwards in time. The two solutions are mirror images of one another, where simply by swapping T^a for T^*, i.e., reversing time's arrow and the expectations assumption, we can recover the exact same solution paths.

The puzzling nature of anticipated monetary policy is transparent here under RE. The effect of an anticipated monetary policy shock on inflation in period T^a is a function of $\Delta_p = T^* - T^a$. As Δ_p increases, inflation in T^a either tends towards zero, remains fixed at $\bar{\imath}$, or diverges to positive/negative infinity depending on the value of $|\phi^{-1}|$. If $|\phi^{-1}| > 1$, then we observe the *forward guidance puzzle*. Specifically, using the terminology of Farhi and Werning (2019), inflation's response to the shock exhibits an anti-horizon effect. Rather than the impact of the shock at the time of the announcement decreasing with the horizon of the policy, the opposite occurs.

The same condition identifies whether inflation will return to steady state under AE following the shock. Therefore, a divergence in inflation in response to the shock under AE diagnoses the anti-horizon effect of inflation, i.e., the forward guidance puzzle, when the same shock is anticipated under RE. We show in the next section that this logic scales to much more complicated and quantitatively relevant macroeconomic models. It requires the more sophisticated machinery of IE-stability analysis to determine the conditions under which the puzzle emerges, but the same intuition of this AE example holds.

Of course, $|\phi^{-1}| < 1$ perfectly coincides with the condition for the existence of a unique REE in this model. Therefore, the determinacy condition is also predictive for the puzzle in this example. However, as we will show, this is just a coincidence. Determinacy or indeterminacy is a misleading criterion by which to assess whether puzzling predictions will occur. For example, consider the case where $\phi = 1$ in the Fisher model. Under this assumption, there is no anti-horizon effect. Forward guidance policy has a fixed impact on inflation in period T^a. The model is indeterminate. But the most extreme predictions of the forward guidance puzzle of an unbounded response as $\Delta_p \to \infty$ are absent. In general, appealing to indeterminacy of an RE equilibrium is only predictive for puzzling responses of endogenous variables to anticipated events under strict assumptions, which IE-stability reveals.

3 Forward guidance in structural models and IE-stability

The connection between IE-stability and the forward guidance puzzle is obtained by studying RE models in the form considered by Evans and Honkapohja (2001). It is straightforward to transfer the insights to more general formulations. To this aim, consider linear structural models that take the following form

$$y_t = \Gamma(\theta) + A(\theta)y_{t-1} + B(\theta)\mathbb{E}_t y_{t+1} + D(\theta)\omega_t \tag{2}$$

$$\omega_t = \rho(\theta)\omega_{t-1} + \varepsilon_t \tag{3}$$

where y_t is a $n \times 1$ vector of endogenous variables with $m \leq n$ jump variables, ω_t is $l \times 1$ vector of exogenous variables, ε_t is a vector of exogenous white noise innovations, and θ is a vector of deep parameters that characterize the behavior of agents and the policymakers.[4] Rearranging Equations (2) and (3)

$$\begin{pmatrix} \mathbb{E}_t y_{t+1} \\ y_t \\ w_t \end{pmatrix} = J + M \begin{pmatrix} y_t \\ y_{t-1} \\ w_{t-1} \end{pmatrix}, \tag{4}$$

it is well-known that a unique bounded solution exists provided that the number of eigenvalues of M that are outside the unit circle equals m. When this condition is satisfied, the unique stable minimum state variable (MSV) solution may be written as

$$y_t = \bar{a}(\theta) + \bar{b}(\theta)y_{t-1} + \bar{c}(\theta)\omega_t, \tag{5}$$

with the solution's dependence on a specific parameterization denoted by θ and which has a non-stochastic steady state denoted by y_{ss}. When this condition is not satisfied such that there are fewer eigenvalues outside the unit circle than m, a continuum of stationary solutions exists of which one class of solutions is given by[5]

$$y_t = \bar{a}(\theta) + \bar{b}(\theta)y_{t-1} + \bar{c}(\theta)\omega_t + \bar{f}(\theta)v_t, \tag{6}$$

where v_t is an arbitrary martingale difference sequence (MDS) and f is an arbitrary matrix that may also depend on θ.

[4] Unless specified, we assume all relevant inverses exist to do the required calculations.

[5] Chapter 10 of Evans and Honkapohja (2001) offers a complete characterization of the full set of sunspot solutions of models of the form (2).

3.1 RE solutions to forward guidance announcements

For the purposes of exploring forward guidance in a general framework, we define a forward guidance announcement, its impact, and the forward guidance puzzle as follows:

Definition: Forward Guidance Announcement *A Forward Guidance Announcement (FGA) is a tuple* $\{\theta_{T^a}, \theta_{T^*}\}$ *such that* $T^* - T^a = \Delta_p > 0$, *where* θ_{T^a} *is the vector of structural parameters that governs the economy from the time of the announcement,* T^a, *until time* $T^* - 1$. θ_{T^*} *is the vector of structural parameters that governs the economy at time* $t \geq T^*$.

A FGA defines a sequence of structural equations

$$y_t = \begin{cases} \Gamma(\theta_{T^a}) + A(\theta_{T^a})y_{t-1} + B(\theta_{T^a})\mathbb{E}_t y_{t+1} + D(\theta_{T^a})\omega_t & \text{if } T^a \leq t < T^* \\ \Gamma(\theta_{T^*}) + A(\theta_{T^*})y_{t-1} + B(\theta_{T^*})\mathbb{E}_t y_{t+1} + D(\theta_{T^*})\omega_t & \text{if } t \geq T^* \end{cases} . \tag{7}$$

Rational solutions to the system of equations are time-varying coefficient analogues to the usual solutions (5) and (6). Unique or determinate FGA solutions take the form of

$$y_t = \bar{a}_t(\theta_t) + \bar{b}_t(\theta_t)y_{t-1} + \bar{c}_t(\theta_t)\omega_t. \tag{8}$$

Indeterminate FGA solutions also have time-varying parameters, and one class of these solutions can be expressed as

$$y_t = \bar{a}_t(\theta_t) + \bar{b}_t(\theta_t)y_{t-1} + \bar{c}_t(\theta_t)\omega_t + \bar{f}_t(\theta_t)v_t \tag{9}$$

where v_t is an arbitrary MDS.

Definition: Impact *The impact of an FGA is defined as* $|y_{ss} - \mathbb{E}[y_{T^a}]|$, *where* $\mathbb{E}[y_{T^a}]$ *is the unconditional expectation of the vector of endogenous variables at time* $t = T^a$ *and* y_{ss} *is the steady state of the model when* $t < T^a$.

Definition: Forward Guidance Puzzle *An FGA* $\{\theta_{T^a}, \theta_{T^*}\}$ *is said to exhibit the forward* *guidance puzzle if its impact is unbounded* ($|y_{ss} - \mathbb{E}[y_{T^a}]| \to \infty$) *as* $\Delta_p \to \infty$.

There is no standard definition for the forward guidance puzzle. Our definition tries to capture three specific aspects of the puzzle in a general way: the effect of policy at the time of the announcement (*impact*), how that policy's impact grows as its implementation date is pushed farther into the future, and that the impact is independent of the realizations of the exogenous shocks. In other words, the effect is not a function of good luck or bad luck, but a direct effect of a credible policy promise. Our definition is agnostic about the form that the policy takes in the model. We are allowing any parameters to change as a result of the policy announcement. We place only two constraints on FGA policies for clarity, which can be generalized.

First, we assume elements of ε_t are not included in the θ_i's. Anticipated shocks such as monetary policy shocks are modeled as temporary changes in the intercept of the policy rule rather than known realizations of ε in the future. Second, FGAs are defined as only consisting of two regimes. Assessing the contemporaneous impact of any FGA can always be cast as a two-period problem. For example, the standard forward guidance thought experiment typically has three parts:

1. a suspension of active policy for some duration Δ_p
2. a one-time anticipated monetary policy shock occurring in period T^*
3. active policy resuming thereafter.

The suspension of active policy, the policy shock (a change in the intercept of the policy rule in our formulation), and the resumption of policy are three separate regimes that can be represented by three different θ_i's. However, what matters for assessing the impact of an FGA is the θ_{T^a} regime and the solution that prevails in period $t = T^*$. If there are additional regimes, that information becomes encoded in the θ_{T^*} solution. Once the solution at time T^* is known, all that matters for assessing the forward guidance puzzle is its relationship to the T^a regime. In this way, more complicated FGAs may be cast into a two-regime problem for the purposes of assessing its impact.

3.1.1 FGA solutions

The most common strategy for obtaining the RE solution to an FGA is to use the method of undetermined coefficients combined with backward induction. This is the approach taken by Eggertsson and Woodford (2003) to study optimal policy at the ZLB. It is also the approach that underpins the solution method for anticipated structural changes in Cagliarini and Kulish (2013) and Kulish and Pagan (2017). Similar approaches are also employed in Cho and Moreno (2011) and to solve for RE solutions of Markov-switching DSGE models as in Baele, Bekaert, Cho, Inghelbrecht and Moreno (2015) and Cho (2016).

Given an FGA $\{\theta_{T^a}, \theta_{T^*}\}$, the method proceeds as follows.[6] First, assume there exists a unique bounded solution for the θ_{T^*} regime when $t \geq T^*$ that takes the form of Equation (5) (the indeterminate case is discussed separately). We can recover this solution from the structural equations (2) by the method of undetermined coefficients, where the expectation of y_{t+1} in time t is given by

$$\mathbb{E}_t y_{t+1} = a + b y_t + c\rho(\theta_{T^*})\omega_t \tag{10}$$

and where a, b, and c are unknown. Substituting equation (10) into equation (2), we have

$$y_t = (I - B(\theta_{T^*})b)^{-1} \left(\Gamma(\theta_{T^*}) + B(\theta_{T^*})a + \left(B(\theta_{T^*})c\rho(\theta_{T^*}) + D(\theta_{T^*})\right)\omega_t + A(\theta_{T^*})y_{t-1}\right). \tag{11}$$

Equating equation (11) with (5), we construct the following equivalences

$$\begin{aligned} a &= (I - B_* b)^{-1} (\Gamma_* + B_* a) && (12) \\ b &= (I - B_* b)^{-1} A_* && (13) \\ c &= (I - B_* b)^{-1} (B_* c \rho_* + D_*)) && (14) \end{aligned}$$

where to simplify notation we write $B_* = B(\theta_{T^*})$ and $B_a = B(\theta_{T^a})$, etc. These equations define the solution for the model when $t \geq T^*$ and the solution can be recovered using any number of standard techniques.

In period $t = T^* - 1$, the MSV solution again takes the same form as equation (5). The

[6] We assume that the inverse of $(I - A(\theta_{T^a})B(\theta_{T^a}))$ exists in order to construct the solution.

expectation of y_{t+1} at time t, however, is no longer unknown. It is given by

$$\mathbb{E}_t y_{t+1} = \bar{a}(\theta_{T^*}) + \bar{b}(\theta_{T^*}) y_t + \bar{c}(\theta_{T^*}) \rho_a \omega_t,$$

where $\bar{a}(\theta_{T^*})$, $\bar{b}(\theta_{T^*})$, and $\bar{c}(\theta_{T^*})$ represent the MSV solution implied by θ_{T^*}. Substituting expectations into equation (2) and equating with equation (5), we now have the following equivalences

$$\begin{aligned}
a &= \left(I - B_a \bar{b}(\theta_{T^*})\right)^{-1} \left(\Gamma_a + B_a \bar{a}(\theta_{T^*})\right) \\
b &= \left(I - B_a \bar{b}(\theta_{T^*})\right)^{-1} A_a \\
c &= \left(I - B_a \bar{b}(\theta_{T^*})\right)^{-1} \left(B_a \bar{c}(\theta_{T^*}) \rho_a + D_a\right),
\end{aligned}$$

which defines the RE solution for $t = T^* - 1$. Continuing to work backwards in time, the RE solution for the FGA, equation (8), may be derived recursively. To illustrate, define j as the number of periods remaining until T^* (i.e. $j = T^* - t$), which allows us to write the RE solution as

$$\bar{a}_j = (I - B_a \bar{b}_{j-1})^{-1} (\Gamma_a + B_a \bar{a}_{j-1}) \tag{15}$$

$$\bar{b}_j = (I - B_a \bar{b}_{j-1})^{-1} A_a \tag{16}$$

$$\bar{c}_j = (I - B_a \bar{b}_{j-1})^{-1} (B_a \bar{c}_{j-1} \rho_a + D_a) \tag{17}$$

where $\bar{a}_0 = \bar{a}(\theta_{T^*})$, $\bar{b}_0 = \bar{b}(\theta_{T^*})$, and $\bar{c}_0 = \bar{a}(\theta_{T^*})$.

From this derivation it is straightforward to arrive at the following proposition.

Proposition 1 *The rational expectations solution for the FGA $\{\theta_{T^a}, \theta_{T^*}\}$ is unique if and only if there is a unique solution to equation (2) for θ_{T^*}.*

The proof is in the appendix and follows Cagliarini and Kulish (2013), who study these solutions in a more general environment. The intuition is that if there is a unique solution for θ_{T^*}, then Equations (15), (16), and (17) trace out a unique trajectory from that starting point.

What if there are multiple solutions for θ_{T^*}? An indeterminate θ_{T^*} regime with n lagged endogenous variables can admit as many as $2n$ choose n MSV solutions. Choosing one of those solutions as a starting point, we can iterate on this mapping to recover the FGA solution associated with that starting point. Additional steps must be taken though if the solutions do not take an MSV form, which we discuss in section 5.

3.2 Connection to IE-stability

Equations (15), (16), and (17) should look familiar to anyone who has studied a model under adaptive learning. This is because under adaptive learning we typically start with the assumption that agents form time t expectations using estimates of the MSV RE solution based on all data up to time $t-1$ such that

$$\mathbb{E}_t y_{t+1} = a_{t-1} + b_{t-1} y_t + c_{t-1} \rho \omega_t.$$

As before, beliefs are substituted into equation (2) to find the actual law of motion for the economy

$$y_t = (I - B b_{t-1})^{-1} \left(\Gamma + B a_{t-1} + (B c_{t-1} \rho + D) \omega_t + A y_{t-1} \right).$$

The actual law of motion reveals the same mapping from agents' beliefs about the MSV RE solution coefficients, $\Phi_{t-1} = (a_{t-1}, b_{t-1}, c_{t-1})$, to the actual equilibrium coefficients as derived previously. This mapping is known as the T-map, where

$$T(\Phi_{t-1}) = \left((I - B b_{t-1})^{-1} (\Gamma + B a_{t-1}), (I - B b_{t-1})^{-1} A, (I - B b_{t-1})^{-1} (B c_{t-1} \rho + D) \right).$$

The T-map summarizes how beliefs map into next period's outcomes, which in turn are used to update beliefs. We can express (15), (16), and (17) equivalently as $\Phi_t = (a_t, b_t, c_t) = T(\Phi_{t-1})$, or:

$$\begin{aligned} a_t &= (I - B b_{t-1})^{-1} (\Gamma + B a_{t-1}) \\ b_t &= (I - B b_{t-1})^{-1} A \\ c_t &= (I - B b_{t-1})^{-1} (B c_{t-1} \rho + D). \end{aligned}$$

Therefore, one may view FGA solutions as iterating on a T-map. The forward guidance puzzle occurs when the T-map is unstable, i.e. when some elements of Φ_t grow without bound as we iterate on the above equations. Diagnosing the forward puzzle is equivalent to studying the stability of an equilibrium of the T-map from a given initial condition.

The T-map is nonlinear. Global stability and instability results for nonlinear difference equations are difficult to obtain without introducing more structure on the problem. In the adaptive learning literature, progress is made by considering local stability. Locally stable T-maps of this form are called Iteratively E-stable (IE-stable):

Definition: IE-Stability *A fixed point of the T-map, $\bar{\Phi}$, is said to be Iteratively E-stable if for all Φ_0 in a neighborhood of $\bar{\Phi}$,*

$$\Phi_N \rightarrow \bar{\Phi}$$

as $N \rightarrow \infty$.

Chapter 5 of Evans and Honkapohja (2001) provides a discussion of the relevant neighborhood and related stability and topological concepts. Evans and Honkapohja (2001) also show that the following conditions determine whether a given $\bar{\Phi}$ is IE-stable.

Theorem 1 (Reformulation of 10.3 Evans and Honkapohja, 2001) *An MSV solution $\bar{a}$, $\bar{b}$, and $\bar{c}$ is IE-stable if all eigenvalues of*

$$\begin{aligned} DT_a(\bar{a}, \bar{b}) &= (I - B\bar{b})^{-1}B \\ DT_b(\bar{b}) &= \left[(I - B\bar{b})^{-1}A\right]' \otimes \left[(I - B\bar{b})^{-1}B\right] \\ DT_c(\bar{b}, \bar{c}) &= \rho' \otimes \left[(I - B\bar{b})^{-1}B\right] \end{aligned}$$

have modulus less than 1. The solution is not IE-stable if any of the eigenvalues have modulus larger than 1.

To translate Theorem 1 to FGA solutions, we have the following proposition.

Proposition 2 *A FGA $\{\theta_{T^a}, \theta_{T^*}\}$ does not exhibit the forward guidance puzzle if*

1. *$\bar{\Phi}(\theta_{T^a})$ exists*

2. *$\bar{\Phi}(\theta_{T^a})$ is IE-stable*

3. *$\Phi_0(\theta_{T^*})$ exists and is in the appropriate neighborhood of $\bar{\Phi}(\theta_{T^a})$*

The proof of the proposition is in the appendix and follows directly from Theorem 1, which leverages well-known stability results for fixed points of dynamic systems.

Proposition 2 provides sufficient conditions which can be used in a wide class of structural models and FGAs to rule out the forward guidance puzzle. The conditions are often necessary and sufficient for applications that study specific forward guidance policies in specific models. The relevant conditions in Proposition 2 for most applications are IE-stability and the notion that the terminal regime solution needs to be close in some sense to $\bar{\Phi}(\theta_{T^a})$. The latter condition arises because the T-map is non-linear, and some applications may involve multiple fixed points. Therefore, failure of the IE-stability condition for some $\bar{\Phi}(\theta_{T^a})$ does not necessarily rule out the existence of a $\Phi_0(\theta_{T^*})$ for which the solution recursion converges to some other stable point.

The appropriate neighborhood condition is not just a technical one. It reveals a useful distinction between announcements that imply modest versus major policy changes in the future. An announcement about a significant future change in the policy framework can imply a terminal regime solution that does not lie in the appropriate neighborhood of a given $\bar{\Phi}(\theta_{T^a})$. For example, if the terminal regime solution adds or decreases the number of state variables on which expectations depend relative to that implied by the announcement regime solution on its own, then the relevant fixed point of the T-map that governs stability may change. Consequently, IE-stability of a specific $\bar{\Phi}(\theta_{T^a})$ may not predict what happens under all feasible FGAs with the same θ_{T^a}. The properties of the model for a specific $\bar{\Phi}(\theta_{T^a})$ regime are predictive for the forward guidance puzzle if the announcement entails modest changes in the policy framework so the terminal regime solution meets condition 3, or $\bar{\Phi}(\theta_{T^a})$ is unique such as when the T-map is linear.

Corollary *When there are no lagged endogenous variables in an economy, an FGA $\{\theta_{T^a}, \theta_{T^*}\}$ does not exhibit the forward guidance puzzle if $\bar{\Phi}(\theta_{T^a})$ exists and is IE-stable.*

Two examples in a Fisher model of inflation concisely illustrate these points:

3.2.1 Example 1 (sufficient vs. necessary and sufficient)

Consider diagnosing the forward guidance puzzle in the Fisher model of section 2,

$$i_t = \mathbb{E}_t \pi_{t+1}$$

for two different FGAs:

$$\text{FGA 1: } i_t = \begin{cases} \phi^a \pi_t & T^a \leq t < T^* \\ \phi^* \pi_t & t = T^* \end{cases} \quad \text{FGA 2: } i_t = \begin{cases} \phi^a \pi_t & T^a \leq t < T^* \\ \bar{i} + \phi^* \pi_t & t = T^* \end{cases}$$

where $\phi^a < 1 < \phi^*$, and $\bar{i} = 0$ for all $t > T^*$. FGA 1 is an announced change in the central bank's reaction function. FGA 2 is an announced change in the reaction function plus a one-time shock ($\bar{i} \neq 0$ only when $t = T^*$). Both FGAs are admissible under our general definition of a forward guidance announcement.

In the absence of stochastic shocks and lagged endogenous state variables, the T-map for both FGAs is given by the following univariate system

$$\bar{a}_j = \frac{1}{\phi^a} \bar{a}_{j-1},$$

where $\bar{a}_0 = 0$ in FGA 1 and $\bar{a}_0 = -\bar{i}/\phi^*$ in FGA 2. Because there are no lagged endogenous variables, the Corollary to Proposition 2 applies and we may study the θ_{T^a} regime in isolation. The T-map has a unique fixed point: $\bar{a}(\theta_{T^a}) = 0$. IE-stability requires $|\phi^a| > 1$, which is not satisfied by assumption. The equilibrium solution for inflation for both FGAs is given in Table 1, with $\bar{i} = 0$ for FGA 1.

For FGA 1, IE-stability is not a necessary condition for ruling out the puzzle. Announced changes to the reaction coefficient on inflation in the interest rate rule do not imply a change in inflation for any Δ_p. The failure of IE-stability in this example is not prescriptive for the forward guidance puzzle. For FGA 2, IE-stability is necessary and sufficient for ruling out and predicting the forward guidance puzzle.

The second FGA example reflects the usual concern that is expressed with the effects of forward guidance. The first FGA example is a special case. Proposition 2 is useful here because it provides the sufficient conditions to rule out the possibility that a specific FGA under study corresponds to a knife edge case. Simulating the first FGA and concluding that the FGA is puzzle free is not a very robust conclusion about forward guidance policy in general in this scenario. Satisfying IE-stability, however, does rule out puzzling behavior for a broader set of forward guidance policies.

3.2.2 Example 2 (appropriate neighborhood of $\bar{\Phi}(\theta_{T^a})$)

To illustrate how the terminal regime matters for determining the puzzle, we now consider a significant change in policy in the Fisher economy, which replicates Cochrane (2017)'s resolution of the puzzle in our framework. There is an implicit assumption in this toy economy that there is a government issuing nominal debt. That nominal debt is what gives rise to the nominal interest rate that the central bank manipulates to implement policy. The Fisher economy is implicitly described by three equations:

$$\begin{aligned} i_t &= \phi\pi_t + \bar{i} \\ i_t &= \mathbb{E}_t\pi_{t+1} \\ b_t &= \delta b_{t-1} + i_t - \beta^{-1}\pi_t, \end{aligned}$$

where the last equation summarizes the government's budget constraint and fiscal policy rule (more detail is given in the Appendix 2). The parameter δ encodes the government's policy towards stabilizing debt. When $\delta < 1$, the government enacts a fiscally responsible, Ricardian policy and debt dynamics are decoupled from inflation dynamics. This is why we can write the equilibrium for inflation without reference to fiscal policy previously. However, that is not the case if the government always engages in a debt-destabilizing or "active" fiscal policy, which implies $\delta > 1$.

Now consider FGAs in this economy when both FGAs include forward guidance about a future one-time interest rate shock when fiscal policy is active and monetary policy is passive in the θ_{T^a} regime. Specifically, assume that the monetary policy reaction function is $0 < \phi = \phi^a < 1$ with $\bar{i} \neq 0$ in $t = T^* - 1$ and otherwise $\bar{i} = 0$, and fiscal policy is $\delta = \delta^a > 1$ for

$T^a \leq t < T^*$. The system for $T^a \leq t < T^*$ can be expressed as:

$$y_t = \Gamma_a + A_a y_{t-1} + B_a \mathbb{E}_t y_{t+1}$$

where $y = (\pi, b)'$ and

$$B_a = \begin{pmatrix} \frac{1}{\phi^a} & 0 \\ 1 - \frac{1}{\beta\phi^a} & 0 \end{pmatrix} \qquad A_a = \begin{pmatrix} 0 & 0 \\ 0 & \delta^a \end{pmatrix}$$

and $\Gamma_a = (-\frac{\bar{i}}{\phi^a}, \frac{\bar{i}}{\beta\phi^a})'$ if $t = T^* - 1$, otherwise $\Gamma_a = 0_{2\times 1}$. Because $\bar{i} = 0$ when $t = T^a$, the θ_{T^a} regime admits two MSV solutions of the form: $y_t = \bar{b}(\theta_{T^a}) y_{t-1}$, where, $\bar{a}(\theta_{T^a}) = 0_{2\times 1}$ and

$$\bar{b}(\theta_{T^a}) = \begin{pmatrix} 0 & \frac{\phi^a - \delta^a}{\phi^a - \beta^{-1}} \\ 0 & \phi^a \end{pmatrix} \text{ or } \tilde{b}(\theta_{T^a}) = \begin{pmatrix} 0 & 0 \\ 0 & \delta^a \end{pmatrix}.$$

One can verify through explicit computation that the largest eigenvalues of the first solution for $DT_a(\bar{a}, \bar{b})$ and $DT_b(\bar{b})$ are $1/\delta^a < 1$ and $\phi^a/\delta^a < 1$, respectively. Hence, $\bar{\Phi}(\theta_{T^a}) = (0_{2\times 1}, \bar{b}(\theta_{T^a}))$ is IE-stable. The IE-stability condition for the second solution is the usual one: $\phi^a > 1$. Therefore, it is not IE-stable.

Which IE-stability condition is predictive for the existence of the forward guidance puzzle? That depends on what is assumed for the θ_{T^*} regime. If $\phi = \phi^* < 1$ and $\delta = \delta^* > 1$ for $t \geq T^*$ is announced at $t = T^a$, then the puzzle is absent. These assumptions select a unique $\Phi_0(\theta_{T^*})$ that is in the appropriate neighborhood of the IE-stable $\bar{\Phi}(\theta_{T^a})$ with active fiscal policy. If instead it is announced that $\phi = \phi^* > 1$ and $\delta = \delta^* < 1$ for $t \geq T^*$, then condition (3) fails for the IE-stable θ_{T^a} solution and the puzzle emerges. We return to the usual Fisher economy studied in example 1.

In both cases there is a unique FGA equilibrium (by Proposition 1) and both economies feature passive monetary policy in the θ_{T^a} regime. The intuition for the different outcomes is that one terminal regime implies history dependence in policy and the other does not. Active fiscal policy implies that inflation always resolves long run fiscal imbalances and therefore equilibrium inflation always depends on the history of debt. Under the alternative assumption, inflation does not play a debt-stabilizing role, as $\delta^* < 1$ implies fiscal policy resolves any fiscal imbalances from T^* on, and $\phi > 1$ implies that inflation is returned to steady state following the completion of the policy. There is no history dependence connecting the terminal regime to the announcement regime, so the solution recursion fails to converge to the IE-stable equilibrium in the limit. In effect, we are pushed back into the world of example 1 despite the temporary detour into active fiscal policy in the θ_{T^a} regime.

The example also highlights a key insight from Proposition 2 that studying the θ_{T^a} regime in isolation is misleading. The θ_{T^a} regime only provides information about announcements involving future regimes that are similar to the regime in place at the time of announcement. Practitioners should expect condition (3) to be satisfied if the θ_{T^a} and θ_{T^*} regimes imply similar economic structures (e.g. the same state variables), otherwise it must be checked.[7] In simple models, checking can be done through direct computations as in this example. In more complicated settings, iterating on the T-map from the initial condition may be required.

3.3 Discussion

Connecting IE-stability to the emergence of the forward guidance puzzle naturally implies the three (not mutually exclusive) categories of resolutions described in the introduction. E- and IE-stability conditions measure the strength of the feedback between expectations and equilibrium outcomes. Reducing expectational feedback by over-discounting the future enhances expectational stability and may – as we show in the next section – eliminate the forward guidance puzzle. Likewise, it is well-known in the learning literature that changing the set of state variables in a model may introduce new fixed points of the T-map with different E-stability properties (see Chapter 8 of Evans and Honkapohja, 2001). Consequently, altering the set of state variables that expectations depend on in equilibrium – "predetermining expectations" – can produce IE-stable fixed points of the T-map that resolve the puzzle. Finally, the learning literature has found that history dependence in policy increases the regions of expectational stability in models (see, for example, Bullard and Mitra, 2007 or Eusepi and Preston, 2011). Any policy that improves expectational stability will have similar effects on mitigating the forward guidance puzzle.

[7] In many applications, including example 2, when $\phi^* > 1 > \delta^*$, $\Phi_0(\theta_{T^*})$ and $\bar{\Phi}(\theta_{T^a})$ imply a change in the set of state variables affecting equilibrium dynamics after T^*. Such applications are more likely to involve violations of condition (3).

4 Application: Over-discounting in a general New Keynesian model

In this section, we show how to use IE-stability to diagnose the forward guidance puzzle in a general New Keynesian environment:

$$x_t = M\mathbb{E}_t x_{t+1} - \sigma\chi\left(i_t - \mathbb{E}_t\pi_{t+1}\right) \tag{18}$$

$$\pi_t = M^f\beta\mathbb{E}_t\pi_{t+1} + \kappa x_t \tag{19}$$

where the usual system of equations is augmented by three additional parameters: M, χ, and M^f to introduce *over-discounting*. These parameters stand in for bounded rationality, heterogeneous agents, or incomplete market assumptions. For example, when $0 < M < 1$, $0 < M^f < 1$, and $\chi = 1$, the model nests the Behavioral New Keynesian (BNK) models of Gabaix (2020). When $0 < M < 1$, $M^f = 1$, and $\chi \neq 1$, the model captures the Tractable Heterogeneous New Keynesian (THANK) assumptions of Bilbiie (2018). A key finding from these modified models is that they solve the forward guidance puzzle under some preferred parameterization assuming that the economy reverts to a Taylor-type rule upon liftoff.

To replicate these papers' findings using IE-stability, consider the following FGA

$$i_t = \begin{cases} 0 & \text{if } T^a \leq t < T^* \\ \bar{i} & \text{if } t = T^* \\ \phi_\pi\pi_t & \text{if } t > T^* \end{cases}, \tag{20}$$

where $\phi_\pi > 1$ to ensure a unique solution. Current inflation targeting implies no history dependence so the Corollary to Proposition 2 applies and we can study the θ_{T^a} regime in isolation. The T-map for the FGA is given by

$$T(\phi) = \Gamma_a + B_a\phi.$$

There exists a unique fixed point of the T-map, $\bar{\Phi}(\theta_{T^a})$, and the relevant eigenvalues governing IE-stability are

$$\lambda_{1,2} = \frac{1}{2}\left(\kappa\sigma\chi \pm \sqrt{(\kappa\sigma\chi + M + \beta M^f)^2 - 4\beta M M^f} + M + \beta M^f\right).$$

IE-stability requires the largest eigenvalue to be less than one in absolute value.

Figure 1 plots the largest eigenvalue identified by IE-stability when M, M^f, χ, or κ are varied. Each line represents a different parameterization of the model. The black line corresponds to the standard model with $M = M^f = \chi = 1$. The red dotted line corresponds to the preferred calibration of Gabaix (2020) with $M = 0.85$, $M^f = 0.8$, and $\chi = 1$. The blue dashed line corresponds to a THANK model with a calibration to match features of the model of McKay et al. (2016) with $M = 0.97$, $M^f = 1$, and $\chi = 0.843$ in the baseline case. Lastly, to capture that M and χ are linked in the THANK model, we solve for the fiscal redistribution that is implied for each considered value of χ, while holding all other parameters constant, and use it to calculate the corresponding M. We then use the χ and the corresponding value of M when solving for the largest eigenvalue in the χ plot. We repeat the same exercise when we vary M to find the implied χ parameter.[8] The remaining parameters are $\beta = 0.99$, $\kappa = 0.11$ and $\sigma = 0.2$ to match Gabaix (2020).

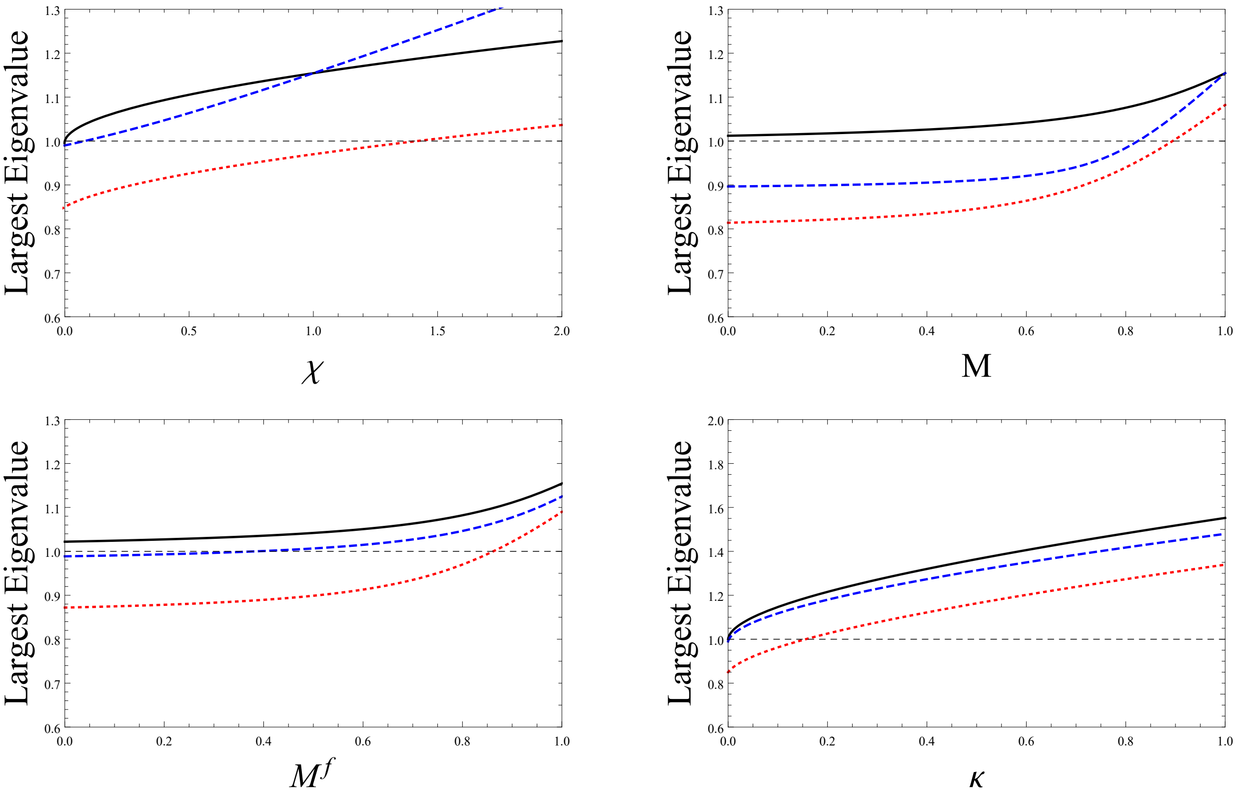
IE-stability reveals three points of interest. First, the forward guidance puzzle is a robust feature of the standard New Keynesian model when the terminal policy regime is a Taylor rule with $\phi_\pi > 1$. Changing a single parameter is insufficient to eliminate the puzzle when the remaining parameters are held at standard values. Second, varying χ and M alone is often not sufficient to rule out the forward guidance puzzle. This result is an example of what Farhi and Werning (2019) call incomplete-markets irrelevance for the puzzle. And finally, all of these forward guidance resolutions are fragile in the sense that there are plausible parameterizations of the THANK and the BNK model that do not solve the puzzle. Resolving the puzzles in both cases is heavily dependent on a flat Phillips curve. Therefore, solutions that rely on these mechanisms are not robust in the sense that they fail to resolve the forward guidance puzzle for some reasonable assumptions about the degree of price rigidity in the economy.

5 Extension I: Indeterminacy and Sunspot FGA solutions

In the discussions up to this point, we have studied unique FGA solutions. Environments in which monetary policy is passive or constrained by the ZLB indefinitely admit multiple

[8]In the THANK model, $M = 1 + (\gamma - 1)\frac{(1-s)}{(1-\gamma\lambda)}$ and $\chi = \frac{1-\lambda}{1-\gamma\lambda}$, where $\lambda = 0.21$ is the proportion of hand-to-mouth consumers in the economy, $s = 0.96$ represents a measure of market incompleteness, and γ represents the degree of fiscal redistribution in the model, which we set to 0.3 when held constant.

Figure 1: IE-stability Exploration



Notes: Largest eigenvalue identified by IE-stability conditions. A value above one indicates the presence of the forward guidance puzzle. The black line shows the standard parameterization of the model, the blue dashed line shows the THANK parameterization of the model, the red dotted line shows BNK parameterization of the model. Specific parameter values for each line are given in the text.

equilibria. In this section, we show how to derive sunspot FGA solutions and assess their IE-stability. Sunspots introduce new state variables on which expectations may depend. Therefore, sunspots can predetermine expectations and resolve the puzzle. We apply our methods to small and medium-scale New Keynesian models to show that sunspot solutions can resolve the forward guidance puzzle.

5.1 Modeling indeterminacy at the ZLB

Indeterminate solutions to the general model given by (2) and (3) have a different functional form than the MSV solution.[9] This requires us to augment our FGA solution technique. To illustrate the complication, consider sunspot solutions of the form (6), which must satisfy the following equivalences

$$(I - B(\theta)b)\,a = \Gamma(\theta) + B(\theta)a \tag{21}$$

$$(I - B(\theta)b)\,b = A(\theta) \tag{22}$$

$$(I - B(\theta)b)\,c = B(\theta)c\rho(\theta) + D(\theta) \tag{23}$$

$$(I - B(\theta)b)\,f = 0_n. \tag{24}$$

A non-zero solution for f implies $(I - B(\theta)b)$ is singular. Inverting the matrices required to construct the recursion studied in the previous section is not feasible.

To overcome this issue, we use the method proposed by Bianchi and Nicolò (2021) (referred to as BN for the remainder of the paper) to augment the model with auxiliary equations that allow us to solve for a specific sunspot solution of our choosing and express the VARMA(p,q) sunspot solutions as a VAR(1) solution with the same reduced form as equation (5). With this reduced form in hand, it is straightforward to produce the same recursions - equations (15), (16), and (17) - studied previously, where we can apply IE-stability.

To this aim, recall that m denotes the number of jump variables in the system of equations under study. Let $n_f < m$ be the number of eigenvalues outside the unit circle recovered from the matrix M in equation (4) such that the model is indeterminate. Define k as the degree

[9]For example, an indeterminate model with VAR(1) MSV solutions admits VARMA(2,1) indeterminate solutions that depend on extraneous MDS sunspot shocks. Evans and McGough (2011) show that sunspot solutions admit both a "general form" representation and a "common factor" representation and it is possible to move between the representations.

of indeterminacy such that $k = m - n_f$. Following BN, we append the following auxiliary equations to the model:

$$s_t = \begin{pmatrix} \frac{1}{\alpha_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\alpha_2} & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & \frac{1}{\alpha_k} \end{pmatrix} s_{t-1} + \begin{pmatrix} x_{1,t} - \mathbb{E}_{t-1}x_{1,t} \\ x_{2,t} - \mathbb{E}_{t-1}x_{2,t} \\ \vdots \\ x_{k,t} - \mathbb{E}_{t-1}x_{k,t} \end{pmatrix} - \begin{pmatrix} v_{1,t} \\ v_{2,t} \\ \vdots \\ v_{k,t} \end{pmatrix} \tag{25}$$

where $|\alpha_i| < 1$ for $i = 1, \dots, k$, $v = (v_1, v_2, \dots, v_k)'$ is the vector of mean-zero iid sunspot shocks, and $x = (x_1, x_2, \dots, x_k)' \subset y$ is the vector of forward looking variables that we choose to follow sunspot processes. The auxiliary equations do not add economic structure to the model. They are simply a device that allows MSV and sunspot solutions to have the same reduced form. By choosing different values for the α_i's either inside or outside of the unit circle, standard RE solution techniques when applied will select either a sunspot or MSV solution, respectively.

Equation (25) is added to equations (2) and (3) to form

$$\tilde{A}_0(\theta)z_t = \tilde{\Gamma}_0(\theta) + \tilde{A}_1(\theta)z_{t-1} + \tilde{B}_0(\theta)\mathbb{E}_t z_{t+1} + \tilde{D}_0(\theta)u_t \tag{26}$$

$$u_t = \tilde{\rho}(\theta)u_{t-1} + \varphi_t \tag{27}$$

where $z_t = (y'_{\not x,t}\ x'_t\ s'_t\ \mathbb{E}_t x'_{t+1})'$, $\varphi_t = (\varepsilon'_t\ v'_t)'$ and

$$\tilde{A}_0(\theta) = \begin{pmatrix} I_{n-k} & 0_{n-k\times k} & 0_{n-k\times k} & 0_{n-k\times k} \\ 0_{k\times n-k} & I_k & 0_{k\times k} & 0_{k\times k} \\ 0_{k\times n-k} & -I_k & I_k & 0_{k\times k} \\ 0_{k\times n-k} & 0_{k\times k} & 0_{k\times k} & I_k \end{pmatrix}, \quad \tilde{\Gamma}_0(\theta) = \begin{pmatrix} \Gamma_y(\theta) \\ \Gamma_x(\theta) \\ 0_{k\times 1} \\ 0_{k\times 1} \end{pmatrix}$$

$$\tilde{A}_1(\theta) = \begin{pmatrix} A_y(\theta) & A_{yx}(\theta) & 0_{n-k\times k} & 0_{n-k\times k} \\ A_{xy}(\theta) & A_x(\theta) & 0_{k\times k} & 0_{k\times k} \\ 0_{k\times n-k} & 0_{k\times k} & \alpha^{-1} & -I_k \\ 0_{k\times n-k} & 0_{k\times k} & 0_{k\times k} & 0_{k\times k} \end{pmatrix}, \quad \tilde{B}_0(\theta) = \begin{pmatrix} B_y(\theta) & B_{yx}(\theta) & 0_{n-k\times k} & 0_{n-k\times k} \\ B_{xy}(\theta) & B_x(\theta) & 0_{k\times k} & 0_{k\times k} \\ 0_{k\times n-k} & 0_{k\times k} & 0_{k\times k} & 0_{k\times k} \\ 0_{k\times n-k} & I_k & 0_{k\times k} & 0_{k\times k} \end{pmatrix}$$

$$\tilde{D}_0(\theta) = \begin{pmatrix} D_y & 0_{k\times k} \\ D_x & 0_{k\times k} \\ 0_{k\times l} & -I_k \\ 0_{k\times l} & 0_{k\times k} \end{pmatrix}, \ \tilde{\rho}(\theta) = \begin{pmatrix} \rho & 0_{l\times k} \\ 0_{k\times l} & 0_{k\times k} \end{pmatrix},$$

where $\alpha^{-1} = diag(1/\alpha_1, \ldots, 1/\alpha_k)$, $A_y(\theta)$ is the $n-k \times n-k$ submatrix of A associated with $y_{\not x} \subset y$, $A_{yx}(\theta)$ is the $n-k \times k$ submatrix of A associated with $x \subset y$, and so on and so forth. Multiplying both sides by $\tilde{A}_0(\theta)^{-1}$ yields the usual reduced form

$$z_t = \tilde{\Gamma}(\theta) + \tilde{A}(\theta)z_{t-1} + \tilde{B}(\theta)\mathbb{E}_t z_{t+1} + \tilde{D}(\theta)u_t \tag{28}$$

Note that (28) collapses to (2) when $k = 0$. When $k > 0$, solutions of these equations take on the usual MSV VAR(1) form in z_t, however, x_t and y_t may now follow high order processes.

To solve for the rational solution of a FGA, $\{\theta_{T^a}, \theta_{T^*}\}$, where θ_{T^*} is indeterminate with k degrees of indeterminacy, we can proceed as follows.

1. Choose k forward looking variables $x \subset y$ to follow sunspots.
2. Append k auxiliary equations (25) to the reduced form.
3. Rearrange the reduced form to arrive at (28).
4. Solve for the sunspot solution implied by θ_{T^*} and your sunspot choices x and v

$$z_t = \tilde{a}(\theta_{T^*}) + \tilde{b}(\theta_{T^*})z_{t-1} + \tilde{c}(\theta_{T^*})u_t. \tag{29}$$

5. Construct the usual recursion (15) - (17) with the augmented model's reduced form.
6. Iterate to find the solution to z_t.

Explicit in this process is that the FGA solution assumes coordination on a sunspot in the terminal regime, θ_{T^*}. Conditioning on a specific sunspot allows us to recover an FGA solution in the exact same way we proceeded in the determinate case. In economic terms, this is akin to the policy announcement coordinating agents' expectations on the sunspot. Once expectations are coordinated on the sunspot, they are no longer forward looking when considering the impact of the FGA.[10]

5.2 Sunspot Solutions of NK Models

To apply the sunspot method to the standard New Keynesian model, consider equations (18) and (19) with $M = M^f = \chi = 1$ and the following policy rule

$$i_t = \phi_\pi \pi_t + \phi_x x_t. \tag{30}$$

We consider the possibility of a sunspot in either inflation or output expectations and append the following equation to the model (18), (19), and (30) to create the augmented system

$$s_t = \frac{1}{\alpha} s_{t-1} - \nu_t + q_t - \mathbb{E}_{t-1} q_t, \tag{31}$$

where ν_t is the sunspot and $q \in \{x, \pi\}$. When $0 < \alpha < 1$, standard RE solution techniques select the sunspot equilibrium. When $\alpha > 1$, the MSV RE solution is selected and the auxiliary process has no impact.

The addition of the auxiliary equation creates an augmented system of the form $\tilde{A}_0 z_t = \tilde{A}_1 z_{t-1} + \tilde{B}_0 \mathbb{E}_t z_{t+1} + \tilde{D}_0 \varepsilon_t$ where $z_t = (x_t, \pi_t, s_t, \mathbb{E}_t \pi_{t+1})'$ if $q = \pi$, or $z_t = (x_t, \pi_t, s_t, \mathbb{E}_t x_{t+1})'$ when $q = x$.[11] The augmented model again has two types of solutions: the MSV solution ($z_t = \bar{b}^1 z_{t-1} + \bar{c}^1 \varepsilon_t$) and a continuum of sunspot solutions of the form: $z_t = \bar{b}^2 z_{t-1} + \bar{c}^2 \varepsilon_t$. The solutions have a closed form. In the case of $q = \pi$, the sunspot solution ($z_t = \bar{b}^2 z_{t-1} + \bar{c}^2 \varepsilon_t$) is

$$\bar{b}^2 = \begin{pmatrix} 0 & 0 & \frac{\Omega - \sqrt{(\Omega-1)^2 - 4\kappa\sigma(\beta\phi_\pi - 1)} - 1}{2\alpha\kappa} & -\frac{\Omega - \sqrt{(\Omega-1)^2 - 4\kappa\sigma(\beta\phi_\pi - 1)} - 1}{2\kappa} \\ 0 & 0 & -\frac{1}{\alpha} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\Omega - \sqrt{(\Omega-1)^2 - 4\kappa\sigma(\beta\phi_\pi - 1)} + 1}{2\alpha\beta} & \frac{\Omega - \sqrt{(\Omega-1)^2 - 4\kappa\sigma(\beta\phi_\pi - 1)} + 1}{2\beta} \end{pmatrix},$$

[10] In the Appendix 2, we demonstrate the method using the same model studied by Lubik and Schorfheide (2003) and BN in their explorations of sunspot solutions to illustrate that indeed this method recovers the same sunspot solutions.

[11] Note that we substituted (30) into (18)-(19) to form the augmented system in this section.

where $\Omega = \sigma\beta\phi_x + \kappa\sigma + \beta$.[12] The relevant IE-stability condition in this case is

$$\frac{2\beta}{\sqrt{(\kappa\sigma+1)^2 - 4\beta\kappa\sigma\phi_\pi + 2\beta(\kappa\sigma-1)\left(\sigma\phi_x+1\right) + \left(\beta(1+\sigma\phi_x)\right)^2} + \beta + \kappa\sigma + \beta\sigma\phi_x + 1} < 1,$$

which is always between zero and one for standard parameter values of κ, σ, and β if $\phi_\pi < \frac{\kappa+(\beta-1)\phi_x}{\kappa}$. The sunspot solution, therefore, satisfies IE-stability for parameter values that are usually predictive of the forward guidance puzzle.

Now consider the following FGA:

$$i_t = \begin{cases} \phi_\pi^a \pi_t + \phi_x^a x_t & \text{if } T^a \leq t < T^* \\ \phi_\pi^a \pi_t + \phi_x^a x_t + \bar{i} & \text{if } t = T^* \\ \phi_\pi^* \pi_t + \phi_x^* x_t & \text{if } t > T^* \end{cases}, \tag{32}$$

which nests the standard ZLB forward guidance thought experiment (20) as a special case.

Proposition 3: *Consider the New Keynesian model and FGA given by (18), (19), (32) with $M = M^f = \chi = 1$ and $\bar{i} \neq 0$:*

1. *The MSV FGA solution ($\alpha > 1$) does not exhibit the forward guidance puzzle if and only if $\bar{\Phi}(\theta_{T^a})$ is IE-stable.*

2. *When $\phi_\pi^a < \frac{\kappa+(\beta-1)\phi_x^a}{\kappa}$, sunspot FGA solutions ($0 < \alpha < 1$) exist that do not exhibit the forward guidance puzzle if and only if*

$$\phi_\pi^* < \frac{\kappa + (\beta-1)\phi_x^*}{\kappa}$$

The proof is in the appendix. The proposition states that IE-stability is a necessary and sufficient condition to rule out the puzzle in the MSV case, but that the sunspot solutions under consideration are not prone to the puzzle. The intuition for why the sunspot studied here resolves the forward guidance puzzle is that coordination on the sunspot pins down inflation or output expectations in the FGA equilibrium and removes the explosive dynamics. For

[12] By rearranging terms in $\bar{b}^2$, we can show that $s_t = 0$ in the sunspot equilibrium. In other words, the appended process is only a tool that selects a sunspot solution of (18), (19), and (30) and s_t never directly impacts equilibrium dynamics of π, x, or i. Also note that the MSV solution for π, i and x is unique, but there are infinitely many sunspot solutions of the form considered in this section (each sunspot solution is indexed by an arbitrary sequence of sunspot shocks, $\{\nu_t\}$).

example, if we choose inflation to follow a sunspot, then the inflation forecast error at the time of announcement is entirely determined by the non-fundamental sunspot variable: $\pi_{T^a} - \mathbb{E}_{T^a-1}\pi_{T^a} = v_t$ in equilibrium, where $s_{T^a-1} = 0$ by construction. Therefore, π_{T^a} is unaffected by the announcement. These puzzle-free equilibria exist so long as the terminal regime of the model is indeterminate. If the terminal regime is determinate, however, then $\Phi_0(\theta_{T^*})$ is the MSV solution, which is not in the appropriate neighborhood of the IE-stable sunspot solutions (condition (3) of Proposition 2 is violated) and the puzzle returns.

5.3 Sunspots in a medium-scale model

The sunspot resolution of the puzzle is not just a feature of the simple model. It scales. To illustrate, we append equation (31) onto the model of Smets and Wouters (2007) and assume $q = \pi$. We then consider the standard forward guidance thought experiment given by (20).

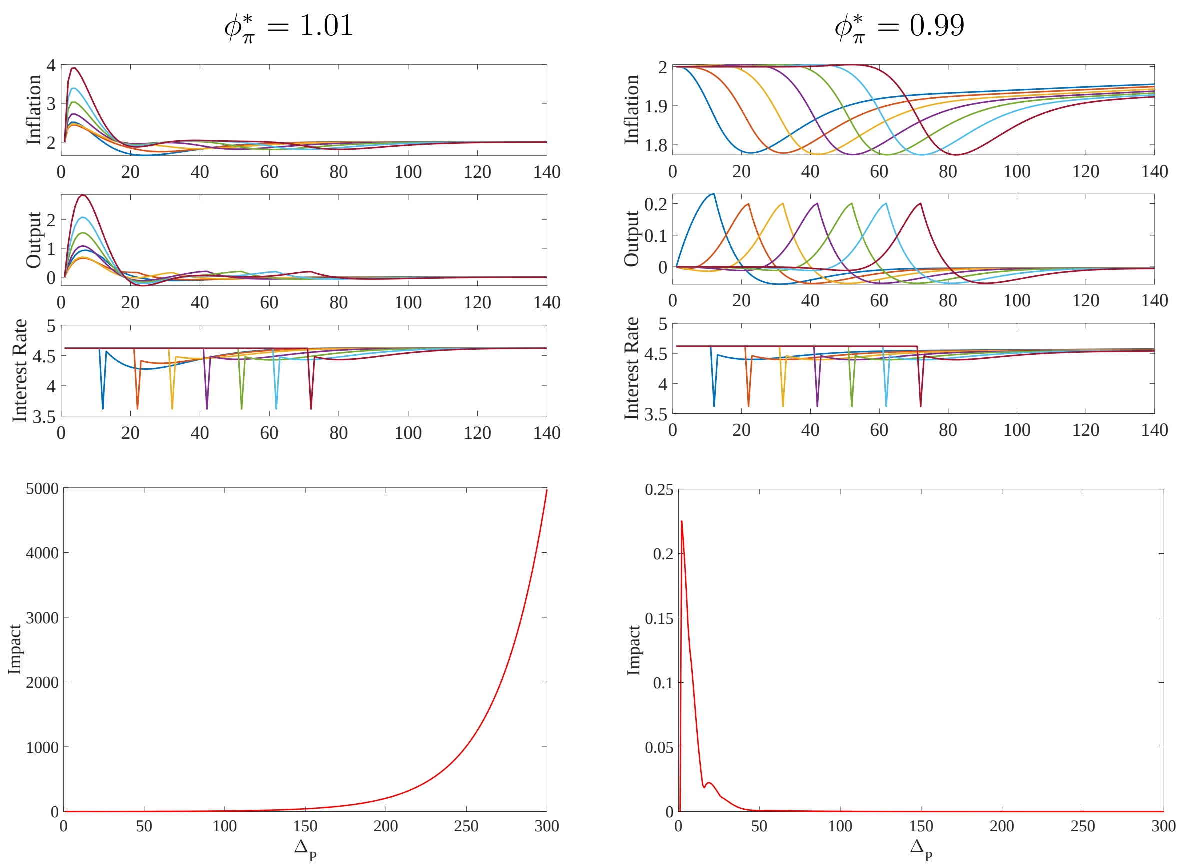
Figure 2 shows the dynamic response of output, inflation, and the interest rate to forward guidance announcements with two different terminal regimes. In the first FGA, we set $\phi_\pi^* = 1.01$ such that the terminal regime is determinate. In the second FGA, $\phi_\pi^* = 0.99$ such that the terminal regime is indeterminate.[13] The θ_{T^a} regime in both cases is the same and features a complete suspension of the monetary policy rule as in (20). The remaining parameters of the model are set to the mean posterior values reported in Smets and Wouters (2007). In the former case, the impact of the policy is increasing in Δ_p. However, when $\phi_\pi^* < 1$ and there is a sunspot solution, there is a well-behaved bounded response.

IE-stability reveals that one explanation for why monetary policy forward guidance may not be arbitrarily powerful is that agents believe that policy will remain passive after liftoff and there is coordination on a sunspot. The notion that forward guidance also conveys a preference for passive policy after liftoff seems plausible. Moreover, it is possible to empirically test this hypothesis using methods similar to those considered in Lubik and Schorfheide (2004) and BN in conjunction with the methods of Kulish and Pagan (2017) and Kulish, Morley and Robinson (2017). This is a compelling case for future research.

IE-stability analysis is also easy to conduct in models of this size. It requires only a few lines

[13] The sunspot shock is parameterized as $s_t = \frac{1}{\phi_\pi} s_{t-1} - \nu_t + \pi_t - \mathbb{E}_{t-1}\pi_t$ in the simulations, where ϕ_π is the response to inflation in the policy rule. For the exercise we chose two calibrations that are on either side of the boundary of indeterminacy ($\phi_\pi = 1$) to show that Proposition 3 generalizes to a medium-scale model. Qualitatively similar results obtain, e.g., under the assumption that the central bank pegs the the interest rate at a steady state value after liftoff ($\phi_\pi = 0$). We shows these results in Appendix 4.

Figure 2: Forward Guidance Announcements in the Smets and Wouter's model



Notes: The top figures shows the paths of output, inflation, and interest rates for anticipated 100-basis point monetary policy shocks with $\Delta_p = 10, 20, \ldots, 70$. The bottom figures show the impact using the L_∞ norm for $\Delta_p = 1, ..., 300$.

of code to check the relevant eigenvalues from matrices that are typically already constructed for any numerical study of these models. We relied here on code provided by Jones (2017) that works with Dynare for modeling the zero lower bound, which extracts the relevant matrices needed for this type of analysis.

6 Extension II: Markov switching FGA Solutions

We now extend our methods to models with Markov-switching policy regimes. Markov-switching models are sometimes used in the empirical literature to study the economy at the ZLB, and in these modeling frameworks the conditions for determinacy and IE-stability of a rational expectations equilibrium may differ substantially.

We provide two examples in this section that showcase the power of IE-stability analysis to determine whether the forward guidance puzzle is present when history dependent policy is introduced. Specifically, we first study Markov-switching between active and passive fiscal policy and active and passive inflation targeting. We then study price level targeting with varying levels of central bank credibility. For the active fiscal policy case, we assume agents contemplate the possibility of either active fiscal policy/passive monetary policy in the terminal regime or vice versa. In this environment, if there is a sufficiently high probability that the economy enters a passive fiscal policy regime, and a correspondingly low probability of an active fiscal policy, then there is indeterminacy when analyzing the ZLB regime in isolation as in the standard model. However, the MSV FGA solution in many cases remains IE-stable. Like in our sunspot example, indeterminacy at the ZLB is not predictive for whether the forward guidance puzzle is present.

In a New Keynesian model of forward guidance with price level targeting after liftoff, we find that the relevant IE-stability eigenvalue is always one. The unit eigenvalue precludes arbitrarily powerful effects of forward guidance as Δ_p is varied, but it also points to the puzzling prediction that a promise made arbitrarily far in the future can have a significant effect on the economy at the time of announcement. We use Markov-switching in this environment to add exogenous credibility considerations to the FGA to see how this affects the prediction. We assume that agents place some probability that the central bank will renege on announced forward guidance policy. We show numerically that if agents attach an arbitrarily small positive probability to the prospect that the central bank reneges on its commitment in the next period, then the relevant IE-stability eigenvalue falls below one and the forward guidance puzzle is totally resolved under price level targeting.

6.1 IE-stability in Markov-switching models

We follow McClung (2021) and consider FGAs in a class of Markov-switching structural models of n equations of the form

$$y_t = \Gamma(\theta,\xi_t) + A(\theta,\xi_t)y_{t-1} + B(\theta,\xi_t)\mathbb{E}_t y_{t+1} + D(\theta,\xi_t)\omega_t \tag{33}$$

$$\omega_t = \rho(\theta,\xi_t)\omega_{t-1} + \varepsilon_t \tag{34}$$

where ξ_t is an S-state exogenous Markov process, and all variables are defined analogous to those in Section 3. Let P denote the transition probability matrix governing the evolution of ξ_t and define $p_{ij} = Pr(\xi_t = j|\xi_{t-1} = i)$ where p_{ij} is the (i, j)-element of P . We assume the model steady state is independent of ξ_t (e.g. the steady state for $t < T^a$, y_{ss}, does not depend on ξ_t). This is a common assumption in the literature, particularly in analyses of regime-switching models of monetary-fiscal policy interactions where recurring changes in the monetary and fiscal policy stance do not impact the steady state.[14]

The only thing that distinguishes (33) from (2) is the regime-switching variable, ξ_t. In this class of models, agents do not know the future path of ξ_t, and therefore ξ_t allows us to model any uncertainty about the economy's future structure that remains after a FGA.[15] As in Section 3, McClung (2021) obtains the RE solution to a FGA by using the method of undetermined coefficients and backward induction (i.e. the approach is a backward application of techniques developed in Cho, 2016). A (MSV) RE solution for $t = T^*$ takes the form of

$$y_t = a(\xi_t) + b(\xi_t)y_{t-1} + c(\xi_t)\omega_t \tag{35}$$

This implies that the expectation of y_{t+1} in time T^* is given by

$$\begin{aligned} \mathbb{E}_t y_{t+1} &= \mathbb{E}_t a(\xi_{t+1}) + \mathbb{E}_t b(\xi_{t+1}) y_t + \mathbb{E}_t c(\xi_{t+1}) \rho(\theta_{T^*}, \xi_{t+1}) \omega_t & (36) \\ &= \sum_{j=1}^{S} p_{\xi_t j} \left(a(j) + b(j) y_t + c(j) \rho(\theta_{T^*}, j) \omega_t \right) & (37) \end{aligned}$$

Define $b = (b(1), \ldots, b(S))$, $\Xi_*(\xi_t, b) = (I - B_*(\xi_t)\mathbb{E}_t b(\xi_{t+1}))$, and $B_*(\xi_t) = B(\theta_{T^*}, \xi_t)$, $B_a(\xi_t) = B(\theta_{T^a}, \xi_t)$, etc. Substituting equation (36) into equation (33) and rearranging yields the following equivalences

$$\begin{aligned} a(\xi_t) &= \Xi_*(\xi_t, b)^{-1} \left(\Gamma_*(\xi_t) + B_*(\xi_t) \mathbb{E}_t a(\xi_{t+1}) \right) & (38) \\ b(\xi_t) &= \Xi_*(\xi_t, b)^{-1} A_*(\xi_t) & (39) \\ c(\xi_t) &= \Xi_*(\xi_t, b)^{-1} \left(B_*(\xi_t) \mathbb{E}_t c(\xi_{t+1}) \rho_*(\xi_{t+1}) + D_*(\xi_t) \right) & (40) \end{aligned}$$

where the MSV RE solution is given by $\bar{a}(\theta_{T^*}, \xi_t)$, $\bar{b}(\theta_{T^*}, \xi_t)$, and $\bar{c}(\theta_{T^*}, \xi_t)$ satisfying equations (38), (39), and (40) for $\xi_t = 1, \ldots, S$.

[14] A growing literature examines models of this form. We will not review the Markov-switching DSGE literature here, but emphasize that Cho (2016, 2021), Maih (2015), Foerster, Rubio-Ramírez, Waggoner and Zha (2016), Farmer, Waggoner and Zha (2009, 2011) and Barthélemy and Marx (2019), among others, develop analytical tools and solution techniques for models of this form. Our approach most closely resembles Cho (2016).

[15] To be precise, agents know the current realization of ξ_t and P.

In period $t = T^* - 1$, the MSV solution again takes the same form as equation (35). Expectations of y_{t+1} at time $t = T^* - 1$, however, are no longer unknown. They are given by

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t \bar{a}(\theta_{T^*}, \xi_{t+1}) + \mathbb{E}_t \bar{b}(\theta_{T^*}, \xi_{t+1}) y_t + \mathbb{E}_t \bar{c}(\theta_{T^*}, \xi_{t+1}) \rho_*(\xi_{t+1}) \omega_t$$

Continuing to work backwards in time, the RE Markov-switching solution for the FGA can be written recursively as

$$\bar{a}_j(\xi_t) = \Xi_a(\xi_t, \bar{b}_{j-1})^{-1} \left(\Gamma_a(\xi_t) + B_a(\xi_t)\mathbb{E}_t \bar{a}_{j-1}(\xi_{t+1})\right) \tag{41}$$

$$\bar{b}_j(\xi_t) = \Xi_a(\xi_t, \bar{b}_{j-1})^{-1} A_a(\xi_t) \tag{42}$$

$$\bar{c}_j(\xi_t) = \Xi_a(\xi_t, \bar{b}_{j-1})^{-1} \left(B_a(\xi_t)\mathbb{E}_t \bar{c}_{j-1}(\xi_{t+1})\rho_{j-1}(\xi_{t+1}) + D_a(\xi_t)\right) \tag{43}$$

where j is defined as before, $\rho_j = \rho_a$ if $j > 0$ and $\rho_j = \rho_*$ if $j = 0$, and $\bar{a}_0 = (\bar{a}(\theta_{T^*}, 1), \ldots, \bar{a}(\theta_{T^*}, S))$, $\bar{b}_0 = (\bar{b}(\theta_{T^*}, 1), \ldots, \bar{b}(\theta_{T^*}, S))$, and $\bar{c}_0 = (\bar{c}(\theta_{T^*}, 1), \ldots, \bar{c}(\theta_{T^*}, S))$.

The T-map difference equations are now (41), (42), and (43). Define $\Phi = (\bar{a}, \bar{b}, \bar{c})$ where $\bar{a} = (\bar{a}(1), \ldots, \bar{a}(S))$, $\bar{b} = (\bar{b}(1), \ldots, \bar{b}(S))$, and $\bar{c} = (\bar{c}(1), \ldots, \bar{c}(S))$. As in Section 3, a fixed point of this map, $\bar{\Phi}$, is said to be IE-stable if for all Φ_0 in an appropriate neighborhood of $\bar{\Phi}$, $\Phi_N \to \bar{\Phi}$ as $N \to \infty$. McClung (2020) provides E-stability conditions, which can be generalized to IE-stability.

Theorem 2 *An MSV solution*
$\bar{a} = (\bar{a}(1), \ldots, \bar{a}(S))$, $\bar{b} = (\bar{b}(1), \ldots, \bar{b}(S))$, *and* $\bar{c} = (\bar{c}(1), \ldots, \bar{c}(S))$ *is IE-stable if all eigenvalues of*

$$DT_a(\bar{a}, \bar{b}) = \left(\oplus_{k=1}^S \left(I - B(k)\sum_{h=1}^S p_{kh}\bar{b}(h)\right)^{-1} B(k)\right)(P \otimes I_n)$$

$$DT_b(\bar{b}) = \left(\oplus_{k=1}^S \bar{b}(k)' \otimes \left(I - B(k)\sum_{h=1}^S p_{kh}\bar{b}(h)\right)^{-1} B(k)\right)(P \otimes I_{n^2})$$

$$DT_c(\bar{b}, \bar{c}) = \sum_{k=1}^S \left(e_k \otimes \begin{pmatrix} p_{k1}\rho(1)' & \ldots & p_{kS}\rho(S)' \end{pmatrix} \otimes \left(\left(I - B(k)\sum_{h=1}^S p_{kh}\bar{b}(h)\right)^{-1} B(k)\right)\right)$$

have modulus less than 1. The solution is not IE-stable if any of the eigenvalues have modulus *larger than 1.*[16]

The IE-stability condition in Theorem 2 is the local stability condition for a given fixed point solution to (41), (42), and (43), as shown in the appendix. If we set $S = 1$, the model (33) becomes (2) and the IE-stability conditions in Theorem 2 become the IE-stability conditions in Theorem 1. Because the definitions in Section 3 extend to (33), and because y_{ss} does not depend on ξ_t, we can extend Proposition 2 to determine when a model of the form (33) predicts a forward guidance puzzle.

Proposition 4 *Consider (33)-(34). A FGA $\{\theta_{T^a}, \theta_{T^*}\}$ does not exhibit the forward guidance puzzle if*

1. $\bar{\Phi}(\theta_{T^a})$ *exists*
2. $\bar{\Phi}(\theta_{T^a})$ *is IE-stable*
3. $\Phi_0(\theta_{T^*})$ *exists and is in the appropriate neighborhood of* $\bar{\Phi}(\theta_{T^a})$

The proof of Proposition 4 is in the Appendix.

6.2 Active Fiscal Policy

To show how fiscal policy considerations can solve the puzzle when interest rates are pegged at the ZLB, we expand on the simple example presented in section 3 and consider a fully-fledged New Keynesian model of monetary-fiscal interactions. Assume that a fiscal authority raises taxes or surpluses, T, and issues nominal debt, B, to finance fiscal deficits (when $T < 0$). Let b_t and τ_t denote the log deviation of the real debt and surpluses. Fiscal policy is then described by

$$b_t = \beta^{-1}(b_{t-1} - \pi_t) + i_t - \tau_t \tag{44}$$

$$\tau_t = \gamma(\xi_t) b_{t-1}. \tag{45}$$

The value of γ in (45) is referred to as the fiscal stance on debt. The influence of γ on debt dynamics is seen by substituting (45) into (44), which yields an autoregressive process for b_t

[16] The sum operator, $\oplus$, is defined such that $\oplus_{k=1}^{S} A(k) = diag(A(1), \ldots, A(S))$ for generic $n \times n$ matrices $(A(1), \ldots, A(S))$, and e_k is a $S \times 1$ vector with 1 in its k-th entry and zeros elsewhere.

with AR(1) parameter equal to $\beta^{-1} - \gamma$, which is equal to δ from the Fisher model of section 3. When γ is high (i.e. values of γ consistent with $|\beta^{-1} - \gamma| < 1$), increases in government debt, b, are fully amortized by increases in current and future fiscal surpluses, and b_t evolves according to a stable autoregressive process. On the other hand, low values of γ (i.e. $|\beta^{-1} - \gamma| > 1$) lead to unstable debt-dynamics, and changes in inflation combined with passive monetary policy are required to stabilize debt in equilibrium. It is this adjustment of inflation to stabilize debt that introduces history dependence to policy, as we emphasized in earlier sections.

When the economy is at the ZLB, it is natural to assume that fiscal policy is active and monetary policy is passive. However, once policy is unconstrained, it is reasonable to expect that monetary policy will be active and fiscal policy will return to a passive stance. Moreover, fiscal policy regimes can evolve over time in democratic countries (e.g., as new Congresses pursue different fiscal policies). To capture how such beliefs in the terminal regime affect the power of forward guidance, we let γ follow an exogenous 2-state Markov process, $\xi_t \in \{M, F\}$, such that $|\beta^{-1} - \gamma(M)| < 1 < |\beta^{-1} - \gamma(F)|$.[17] We also assume that the monetary regime is characterized by a time-varying Taylor rule of the form:

$$i_t = \phi_\pi(\xi_t)\pi_t \tag{46}$$

where the restriction $\phi_\pi(F) < 1 < \phi_\pi(M)$ is imposed in θ_{T^*}. We impose this last parameter restriction because passive monetary policy (i.e. $\phi_\pi(F) < 1$) allows for debt-stabilizing inflation to occur when $1 < |\beta^{-1} - \gamma(F)|$, whereas active monetary policy (i.e. $1 < \phi_\pi(M)$) helps to prevent coordination on sunspots during periods where debt is being stabilized by fiscal policy (i.e. $|\beta^{-1} - \gamma(M)| < 1$). The full model is given by (18)-(19) and (44)-(46).[18]

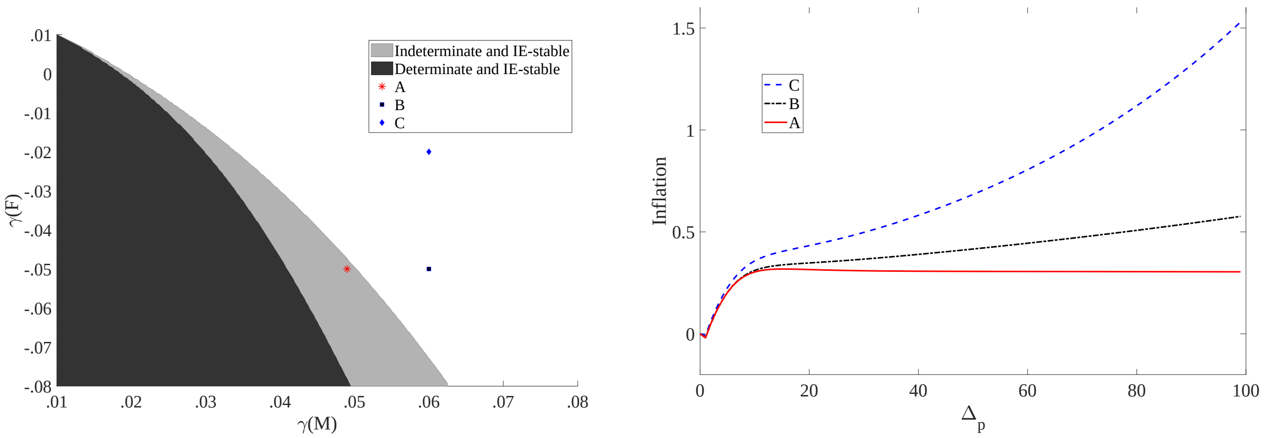
To assess the IE-stability properties, we use numerical techniques and the conditions presented by Theorem 2. Figure 3 shows determinacy and IE-stability regions in the fiscal policy parameter space under an interest rate peg, and it can be seen that IE-stability is satisfied provided that the overall fiscal stance is sufficiently active (i.e. γ is on average relatively small).[19]

[17] Related models are studied by Davig and Leeper (2011), Bianchi and Ilut (2017), Bianchi and Melosi (2017), and Ascari, Florio and Gobbi (2020), among others.

[18] McClung (2021) uses a similar model to show when the forward guidance puzzle is ameliorated by fiscal policy considerations. We refer interested readers to McClung (2021) for more information on the effects of forward guidance under active or non-Ricardian fiscal regimes. We set $\chi = M = M^f = 1$ for this exercise so the model is equivalent to the standard New Keynesian representative agent benchmark.

[19] Note that the left panel of Figure 3 considers a region of the parameter space that allows for $\gamma(F) < 0$.

Figure 3: IE-stability and Determinacy Regions



Notes: The white region is the indeterminate and IE-unstable parameter region. The right figure shows the initial responses of annual inflation (i.e. "1" is 1% annual inflation) in the Markov-switching New Keynesian model for anticipated changes in the interest rate that occur Δ_p periods in the future.

Notice that the indeterminacy and IE-stability conditions are distinct. To show also that the forward guidance puzzle is independent of indeterminacy, we examine three parameterizations from Figure 3. Parameterization A delivers indeterminacy under an interest rate peg, but the corresponding MSV solution is IE-stable. Parameterization B constitutes a small deviation from Parameterization A, and it is IE-unstable under the peg. Parameterization C is still further in the indeterminacy/IE-unstable region of the policy parameter space.[20]

According to our IE-stability criterion, parameterization A should correspond to a puzzle-proof case, whereas parameterizations B and C should not. The initial responses of inflation to anticipated policy shocks for increasing Δ_p's for the three parameterizations confirms the prediction.[21]

Though Davig and Leeper (2011) estimated $\gamma(F) < 0$, many papers in the literature impose the restriction: $\gamma(F) \geq 0$. From Figure 3, it is evident that our main result holds for $\gamma(F) \geq 0$ as there is a region of the parameter space for which $\gamma(F) \geq 0$, the model is indeterminate and an IE-stable solution exists.

[20]We can construct stable common-factor sunspot solutions for each of three parameterizations considered in this section. Hence, multiple equilibria exist under Parameterizations A, B, C. See Cho (2016, 2021) for more information.

[21]In these plots we assume that the economy is in Regime M at the time of the forward guidance announcement, though, qualitatively similar impulse responses can be obtained under the alternative assumption that the economy is in Regime F.

6.3 PRICE LEVEL TARGETING

Returning again to the standard New Keynesian model given by (18) and (19) with $M = M^f = \chi = 1$, we consider the following forward guidance policy:

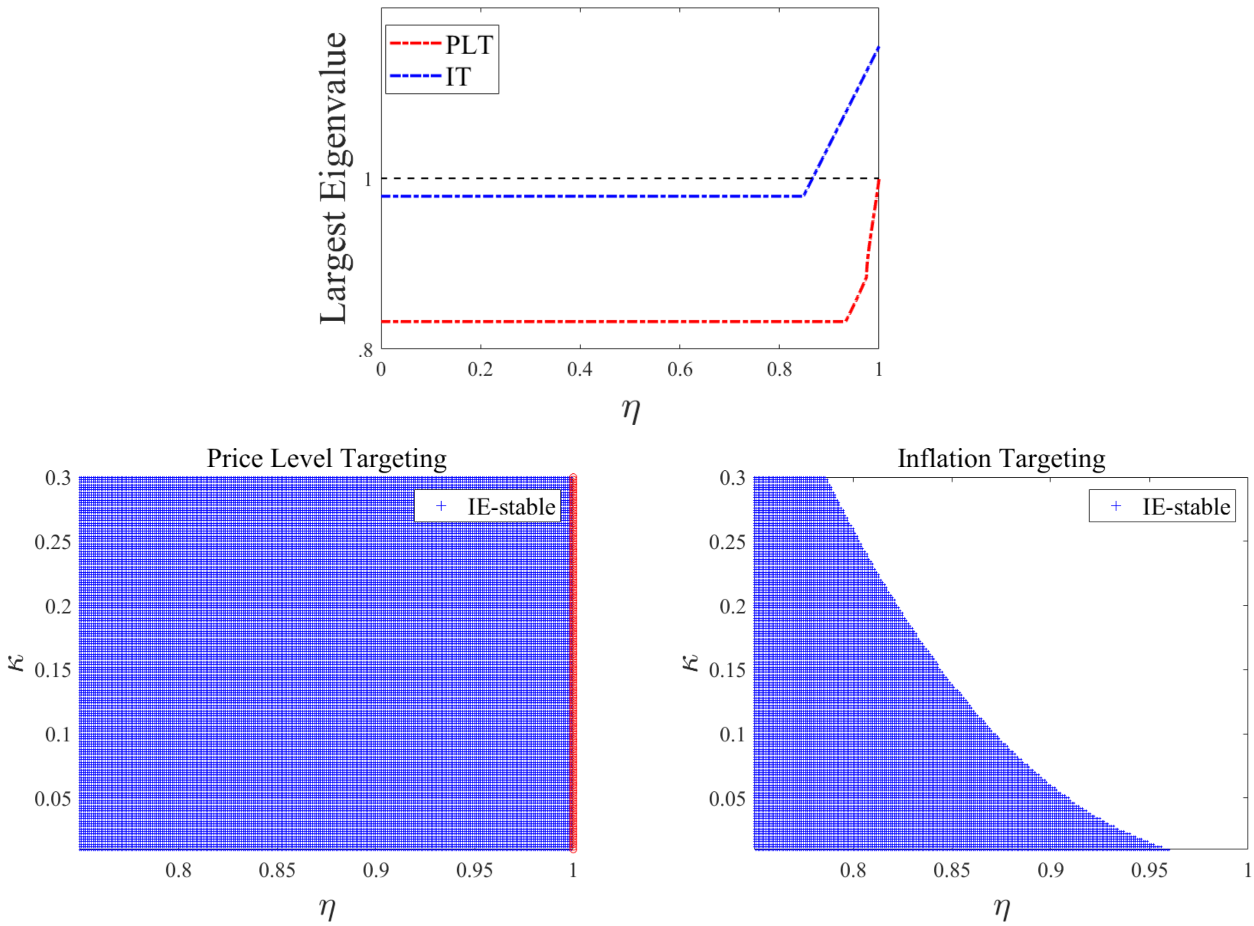
$$i_t = \begin{cases} 0 & \text{if } T^a \leq t < T^* \\ \bar{i} & \text{if } t = T^* \\ \phi_p(p_t - \bar{p}) & \text{if } t > T^* \end{cases},$$

where $\phi_p > 0$, p_t is the (log) of the price level, and $\bar{p}$ is the price level target. Further, consider that the policy announcement might not be viewed as credible. Specifically, assume that agents attach a constant probability, $1 - \eta$, in each period $T^a \leq t \leq T^*$ that the central bank will renege on the promised pegged interest rate of $i_t = 0$ and adjust rates in accordance with the price level targeting rule that governs policy with certainty from period $t > T^*$. When $\eta = 1$, the policy is perfectly credible.

The first panel of Figure 4 traces out the largest eigenvalue identified by IE-stability as a function of η. A fully credible FGA combined with expectations of price level targeting after liftoff generates a unit eigenvalue. This implies that expectations of price level targeting will bound the effects of fully credible forward guidance announcements, but that an anticipated rate cut expected to occur in the infinite horizon has non-zero effects on inflation and output. On the other hand, imperfect credibility (i.e. *any* $\eta < 1$), brings the relevant eigenvalue inside the unit circle. Hence, expectations of price level targeting ensure that (nearly) fully credible forward guidance announcements have bounded effects on inflation and output which go to zero as the anticipated rate change is pushed into the infinite horizon. For comparison, the relevant eigenvalues are plotted from a model of forward guidance which assumes that the central bank reverts to an inflation targeting Taylor rule after liftoff (i.e. the interest rate rule after liftoff is given by (30), as opposed to the price level targeting rule, but all other details of the exercise are the same). For high degrees of credibility, the IE-stability eigenvalue is strictly outside the unit circle and therefore the model is susceptible to the forward guidance puzzle, as we have shown in earlier sections. Only when central bank credibility is low will forward guidance announcements have bounded effects in the model with an inflation targeting central bank.

The second panel shows the relationship of this result with κ, the slope of the Phillips curve.

Figure 4: IE-stability, Price Level Targeting, and Reneging

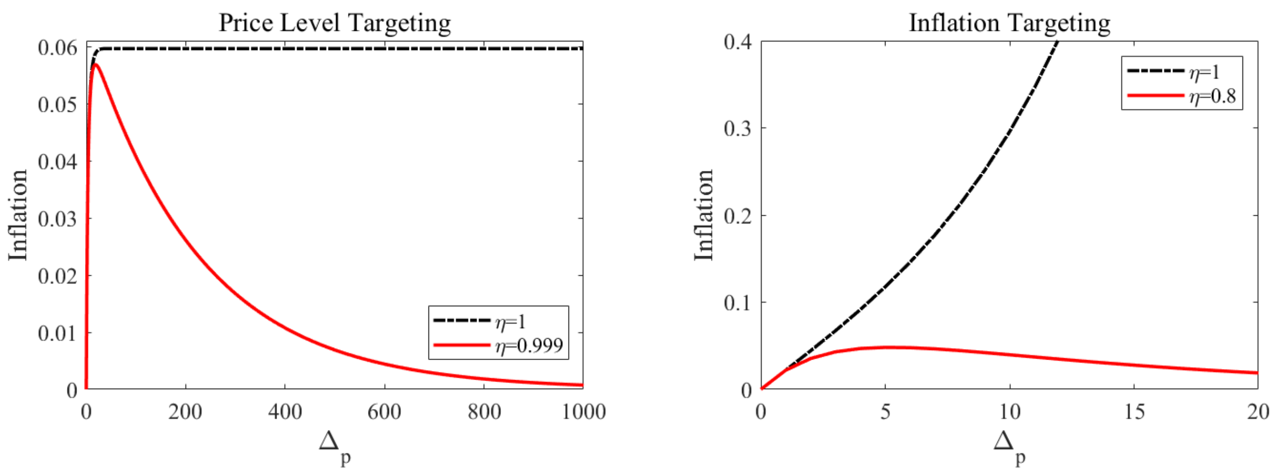


Notes: In the bottom panels, the white region is the IE-unstable parameter region. The red region depicts calibrations that give a unit IE-stability eigenvalue. See section 4 for calibration details (additionally, $\phi_\pi = \phi_p = 1.5$ and $\phi_y = 0$).

Recall that previously, κ was shown to be a key determinate of the puzzle's resolution when relying on dampening the general equilibrium effects of an FGA. There is no such relationship when assuming price level targeting as the terminal regime. In this sense a history dependent price level targeting policy provides a more robust solution of the puzzle. The final panel illustrates the point by showing the same IE-stability graph assuming inflation targeting. Here there is clear relationship with κ, where progressively less credibility is required as κ increases in order to resolve the forward guidance puzzle under an inflation targeting regime.

Figure 5 compares the impact effect under the two policies for different parameterizations of credibility. Perfect credibility under price level targeting bounds the effect but does not reduce the impact of promises made arbitrarily far in the future. Moving away from perfect credibility, however, overturns this prediction.

Figure 5: Reneging and the Forward Guidance Puzzle



Notes: The initial responses of annual inflation (i.e. "1" is 1% annual inflation) for anticipated changes in the interest rate that occur Δ_p periods in the future. See section 4 for calibration details (additionally, $\phi_\pi = \phi_p = 1.5$ and $\phi_y = 0$).

7 Conclusion

We show that IE-stability is a sufficient condition for ruling out the forward guidance puzzle, which we defined in a general way for a wide class of anticipated policy actions in a broad class of models. We furthermore establish that IE-stability is necessary and sufficient for resolving the classic New Keynesian forward guidance puzzle, which is present if announcements about future interest rate changes have unbounded effects on the economy as the timing of the anticipated rate change is pushed into the infinite future.

By establishing the link between IE-stability and the forward guidance puzzle, we put forward three (not mutually exclusive) categories of forward guidance resolutions. The forward guidance puzzle may be eliminated by introducing over-discounting of expectations relative to the baseline model; new assumptions may be added that link expectations to additional state variables in equilibrium, which we call predetermining expectations; or policy may be pursued that implies history dependence. The latter resolution also introduces new state variables which expectations depend on in equilibrium. We distinguish it from predetermining expectations because it is explicitly a policy choice and not an emergent feature of some other aspect of the economy. We believe these categories may help economists anticipate why the puzzle may arise in their model and how it can be mitigated.

There are many extensions beyond the scope of this paper. For example, we show how a single quantitative model can nest puzzling and non-puzzling behavior, which opens the possibility for empirical tests of the power of forward guidance. We also limit our study to linear frameworks, even though many policy relevant studies work with nonlinear models. Many studies of E-stability in these environments are well-established in the literature. Extending our framework to these cases is an interesting avenue for future research.

Appendix

A1 Proofs of propositions

Theorems 1 and 2 in the paper are known results in the literature that we use to prove the key propositions. The relevant citations for where the proofs are found are provided in the main text.

Proof of Proposition 1 Following Cagliarini and Kulish (2013) a straightforward way to proceed is to note that the system of equations given by (7) can be written as

$$\begin{pmatrix} I_n & -B(\theta_{T^a}) & \dots & \dots & 0_n \\ -A(\theta_{T^a}) & I_n & -B(\theta_{T^a}) & \ddots & \vdots \\ 0_n & -A(\theta_{T^a}) & I_n & -B(\theta_{T^a}) & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0_n \\ 0_n & \dots & -A(\theta_{T^a}) & I_n & -B(\theta_{T^a}) \\ 0_n & \dots & \dots & -\bar{b}(\theta_{T^*}) & I_n \end{pmatrix} \begin{pmatrix} y_{T^a} \\ \mathbb{E}_t y_{T^a+1} \\ \vdots \\ \mathbb{E}_t y_{T^*} \end{pmatrix} =$$

$$\begin{pmatrix} \Gamma(\theta_{T^a}) + A(\theta_{T^a}) y_{T^a-1} + D(\theta_{T^a}) \omega_{T^a} \\ \mathbb{E}_t \left(\Gamma(\theta_{T^a}) + D(\theta_{T^a}) \omega_{T^a+1} \right) \\ \vdots \\ \mathbb{E}_t \left(\bar{a}(\theta_{T^*}) + \bar{c}(\theta_{T^*}) \omega_{T^*} \right) \end{pmatrix}$$

The system has $(\Delta_p+1)\times n$ equations and $(\Delta_p+1)\times n$ unknowns given by the vector representing the path of y, $Gy = R$.

Case 1: Suppose that y is unique but θ_{T^*} implies an indeterminate solution. Then $\mathbb{E}_t y_{T^*}$ is not unique. Therefore, y is not unique.

Case 2: Suppose that θ_{T^*} implies a determined solution for y_{T^*}. Then, we can solve this system by Block Trangularization. We assume that $(I - A(\theta_{T^a})B(\theta_{T^a}))^{-1}$ exists throughout our derivations. This is stated in footnote 6 of the main text.

Transform $Gy = R$ to $\mathbb{G}y = P$

$$\begin{pmatrix} I_n & G^{(1)} & \dots & \dots & 0_n \\ 0_n & I_n & G^{(2)} & \ddots & \vdots \\ 0_n & 0_n & I_n & G^{(3)} & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0_n \\ 0_n & \dots & 0_n & I_n & G^{(\Delta_p)} \\ 0_n & \dots & \dots & 0_n & I_n \end{pmatrix} \begin{pmatrix} y_{T^a} \\ \mathbb{E}_t y_{T^a+1} \\ \vdots \\ \mathbb{E}_t y_{T^*} \end{pmatrix} = \begin{pmatrix} P^{(1)} \\ P^{(2)} \\ \vdots \\ P^{(\Delta_p+1)} \end{pmatrix}$$

$$G^1 = -B(\theta_{T^a})$$

$$G^{(k)} = -\left(I + A(\theta_{T^a})G^{(k-1)}\right)^{-1} B(\theta_{T^a}) \text{ for } k = 2, ..., (\Delta_p)$$

$$P^1 = \Gamma(\theta_{T^a}) + A(\theta_{T^a})y_{T^a-1} + D(\theta_{T^a})\omega_{T^a}$$

$$P^{(k)} = \left(I + A(\theta_{T^a})G^{(k-1)})\right)^{-1} (R^{(k)} + A(\theta_{T^a})P^{(k-1)}) \text{ for } k = 2, ..., (\Delta_p)$$

$$P^{(\Delta_p+1)} = (I + \bar{b}(\theta_{T^*})G^{(\Delta_p)})^{-1}(R^{(\Delta_p)} + \bar{b}(\theta_{T^*})P^{(\Delta_p)})$$

$\mathbb{G}$ is upper triangular with ones along the entire diagonal, therefore, it is invertible. The unique path of y is $y = \mathbb{G}^{-1}P$. □

PROOF OF PROPOSITION 2 We can write the impact of an FGA $\{\theta_i\}_{i=T^a,T^*}$ using equations (15), (16), and (17) as

$$\begin{aligned} |y_{ss} - \mathbb{E}\left[y_{T^a}\right]| &= |y_{ss} - \mathbb{E}\left[\bar{a}_j + \bar{c}_j\omega_{T^a} + \bar{b}_j y_{T^a-1}\right]| \\ &= |y_{ss} - \bar{a}_j - \bar{b}_j y_{ss}| \\ &= \left|(I - \bar{b}_j)y_{ss} - \bar{a}_j\right|, \end{aligned}$$

where $|\cdot|$ is any p-norm. By the triangle inequality, it follows that

$$|(I - \bar{b}_j)y_{ss}| + |\bar{a}_j| \geq \left|(I - \bar{b}_j)y_{ss} - \bar{a}_j\right|.$$

If all three conditions are satisfied, then the recursion given by equations (15) - (17) are well defined with initial conditions $\bar{a}_0(\theta_{T^*})$ and $\bar{b}_0(\theta_{T^*})$, and $\bar{a}_j \to \bar{a}(\theta_{T^a})$ and $\bar{b}_j \to \bar{b}(\theta_{T^a})$ as $j = \Delta_p \to \infty$ by Theorem 1. Therefore, in the limit as Δ_p goes to infinity the impact of the FGA can be no larger in magnitude than $|(I - \bar{b}(\theta_{T^a}))y_{ss}| + |\bar{a}(\theta_{T^a})|$. $\square$

Proof of Proposition 3

Part 1: For ease of notation, let $\phi_\pi^a = \phi_\pi$ and $\phi_x^a = \phi_x$. The MSV solution for $t > T^*$ can be expressed as $y_t = (x_t, \pi_t)' = 0_{2\times 1}$. Therefore, the $t = T^a$ solution is given by:

$$\begin{aligned} y_{T^a} &= G_a^{\Delta_p}\Gamma \\ &= Q^{-1}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^{\Delta_p} Q\Gamma \end{aligned}$$

where

$$G_a = \begin{pmatrix} \frac{1}{\kappa\sigma\phi_\pi+\sigma\phi_x+1} & \frac{\sigma-\beta\sigma\phi_\pi}{\kappa\sigma\phi_\pi+\sigma\phi_x+1} \\ \frac{\kappa}{\kappa\sigma\phi_\pi+\sigma\phi_x+1} & \frac{\beta\sigma\phi_x+\beta+\kappa\sigma}{\kappa\sigma\phi_\pi+\sigma\phi_x+1} \end{pmatrix} \qquad \Gamma = \begin{pmatrix} -\frac{\bar{i}\sigma}{\kappa\sigma\phi_\pi+\sigma\phi_x+1} \\ -\frac{\bar{i}\kappa\sigma}{\kappa\sigma\phi_\pi+\sigma\phi_x+1} \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & \frac{-\sqrt{\beta^2\sigma^2\phi_x^2+2\beta^2\sigma\phi_x+\beta^2+2\beta\kappa\sigma^2\phi_x-4\beta\kappa\sigma\phi_\pi+2\beta\kappa\sigma-2\beta\sigma\phi_x-2\beta+\kappa^2\sigma^2+2\kappa\sigma+1}+\beta\sigma\phi_x+\beta+\kappa\sigma-1}{2\kappa} \\ 1 & \frac{\sqrt{\beta^2\sigma^2\phi_x^2+2\beta^2\sigma\phi_x+\beta^2+2\beta\kappa\sigma^2\phi_x-4\beta\kappa\sigma\phi_\pi+2\beta\kappa\sigma-2\beta\sigma\phi_x-2\beta+\kappa^2\sigma^2+2\kappa\sigma+1}+\beta\sigma\phi_x+\beta+\kappa\sigma-1}{2\kappa} \end{pmatrix}$$

and $\Delta_p = T^* - T^a$, $\lambda_1 = \frac{\beta\sigma\phi_x+\beta+\kappa\sigma+1-\sqrt{(-\beta\sigma\phi_x-\beta-\kappa\sigma-1)^2-4(\beta\kappa\sigma\phi_\pi+\beta\sigma\phi_x+\beta)}}{2(\kappa\sigma\phi_\pi+\sigma\phi_x+1)}$, and $\lambda_2 = \frac{\beta\sigma\phi_x+\beta+\kappa\sigma+1+\sqrt{(-\beta\sigma\phi_x-\beta-\kappa\sigma-1)^2-4(\beta\kappa\sigma\phi_\pi+\beta\sigma\phi_x+\beta)}}{2(\kappa\sigma\phi_\pi+\sigma\phi_x+1)}$. This yields the following solution for inflation at the time of announcement:

$$\pi_{T^a} = \bar{i}\left(\mu_1\lambda_1^{\Delta_p} - \mu_2\lambda_2^{\Delta_p}\right)$$

where $\mu_1 = \frac{\kappa\sigma\left(-\sqrt{2\beta(\kappa\sigma(\sigma\phi_x-2\phi_\pi+1)-\sigma\phi_x-1)+(\beta\sigma\phi_x+\beta)^2+(\kappa\sigma+1)^2}+\beta\sigma\phi_x+\beta+\kappa\sigma+1\right)}{2(\kappa\sigma\phi_\pi+\sigma\phi_x+1)\sqrt{2\beta(\kappa\sigma(\sigma\phi_x-2\phi_\pi+1)-\sigma\phi_x-1)+(\beta\sigma\phi_x+\beta)^2+(\kappa\sigma+1)^2}}$ and $\mu_2 = \frac{\kappa\sigma\left(\sqrt{2\beta(\kappa\sigma(\sigma\phi_x-2\phi_\pi+1)-\sigma\phi_x-1)+(\beta\sigma\phi_x+\beta)^2+(\kappa\sigma+1)^2}+\beta\sigma\phi_x+\beta+\kappa\sigma+1\right)}{2(\kappa\sigma\phi_\pi+\sigma\phi_x+1)\sqrt{2\beta(\kappa\sigma(\sigma\phi_x-2\phi_\pi+1)-\sigma\phi_x-1)+(\beta\sigma\phi_x+\beta)^2+(\kappa\sigma+1)^2}} > 0$. If the MSV $\bar{\Phi}(\theta_{T^a})$ is IE-unstable ($\phi_\pi < \frac{\kappa+(\beta-1)\phi_x}{\kappa}$) then $|\lambda_1| < 1 < \lambda_2$ and therefore $\lambda_1^{\Delta_p} \to 0$, $\lambda_2^{\Delta_p} \to +\infty$ and $\pi_{T^a} \to -\infty$ as $\Delta_p \to \infty$ if $\bar{i} > 0$ and $\pi_{T^a} \to \infty$ as $\Delta_p \to \infty$ if $\bar{i} < 0$.

If, however, we have IE-stability ($\phi_\pi > \frac{\kappa+(\beta-1)\phi_x}{\kappa}$), then by Proposition 2 and its Corollary there is no forward guidance puzzle in the MSV case.

IE-stability is necessary and sufficient for the puzzle to be absent in the MSV FGA solution for the FGA (32).

Part 2: For the model under consideration, the recursion (15) - (17) augmented with equation (29) reduces to a one-dimensional difference equation. The difference equation has a stable equilibrium point for which all sunspot solutions that satisfy $\phi_\pi^* < \frac{\kappa+(\beta-1)\phi_x^*}{\kappa}$ are in the stable set of a unique $\bar{\Phi}(\theta_{T^a})$, which is IE-stable.

If $\phi_\pi^* < \frac{\kappa+(\beta-1)\phi_x^*}{\kappa}$ is not satisfied, then part (3) of Proposition 2 fails because the terminal regime is the MSV solution and that solution is unique as shown in Proposition 1. The conclusions of part 1 then follow.

If $\phi_\pi^* < \frac{\kappa+(\beta-1)\phi_x^*}{\kappa}$ is satisfied, then there are two cases to consider. First, consider the case that the sunspot is in the output gap expectations such that $q = x$. The terminal sunspot solution for $t > T^*$ is given by

$$\bar{b}_{T^*} = \begin{pmatrix} 0 & 0 & -\frac{1}{\alpha} & 1 \\ 0 & 0 & -\frac{\psi_{T^*}}{\alpha} & \psi_{T^*} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{G(\psi_{T^*})}{\alpha} & G(\psi_{T^*}) \end{pmatrix}$$

where $\psi_{T^*} = -\frac{\sqrt{(\beta\sigma\phi_x^*+\beta+\kappa\sigma-1)^2-4\kappa\sigma(\beta\phi_\pi^*-1)}+\beta\sigma\phi_x^*+\beta+\kappa\sigma-1}{2\sigma(\beta\phi_\pi^*-1)}$ and $G(\psi_{T^*}) = \frac{1+\sigma\phi_x^*+\sigma\phi_\pi^*\psi_{T^*}}{1+\sigma\psi_{T^*}}$. Note that $\psi_{T^*} > 0$ if $\phi_\pi^* < \frac{\kappa+(\beta-1)\phi_x^*}{\kappa}$ and $\kappa > 0$, $\phi_x^* \geq 0$, and $0 < \beta < 1$. Through direct computation, one can verify that

$$\tilde{b}_j = \begin{pmatrix} 0 & 0 & -\frac{1}{\alpha} & 1 \\ 0 & 0 & -\frac{\psi_j}{\alpha} & \psi_j \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{G_a(\psi_{j-1})}{\alpha} & G_a(\psi_{j-1}) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{1}{\alpha} & 1 \\ 0 & 0 & -\frac{f(\psi_{j-1})}{\alpha} & f(\psi_{j-1}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{G_a(\psi_{j-1})}{\alpha} & G_a(\psi_{j-1}) \end{pmatrix}$$

given $\tilde{b}_0 = \bar{b}_{T^*}$ where $G_a(\psi_{j-1}) = \frac{1+\kappa\sigma\phi_\pi^a+\sigma\phi_x^a}{1+\psi_{j-1}\sigma(1-\beta\phi_\pi^a)}$ and $\psi_j = f(\psi_{j-1}) = \frac{\kappa+\psi_{j-1}(\beta\sigma\phi_x^a+\beta+\kappa\sigma)}{\psi_{j-1}(\sigma-\beta\sigma\phi_\pi^a)+1}$. Therefore, if for all $\psi_{T^*} > 0$ the $\lim_{j\to\infty} \psi_j = f(\psi_{j-1}) = \psi_A$, then we have a unique $\bar{\phi}(\theta_{T^a})$ solution for which we can assess IE-stability.

To show ψ_A is the unique rest point for any initial ψ_{T^*}, we note that $f(\psi) - \psi = 0$ is quadratic with two roots $\psi_L < 0 < \psi_A$. Define $\psi_B = -1/(\sigma - \beta\sigma\phi_\pi^a) < \psi_L < 0$. One can show that for all $\psi > \psi_B$ it follows that $f(\psi) < \psi$ as $\psi \to \psi_B^+$, $f(0) = \kappa > 0$, and $lim_{\psi\to\infty} f(\psi) = \bar{f}$ is finite so that $0 < f'(\psi_A) < 1 < f'(\psi_L)$. Therefore, $\psi_j \to \psi_A = -\frac{\sqrt{(\beta\sigma\phi_x^a+\beta+\kappa\sigma-1)^2-4\kappa\sigma(\beta\phi_\pi^a-1)}+\beta\sigma\phi_x^a+\beta+\kappa\sigma-1}{2\sigma(\beta\phi_\pi^a-1)}$ given $\psi_0 = \psi_{T^*} > 0$.

We now assess IE-stability, which obtains eigenvalues of $F_A = (I - \tilde{B}_a\tilde{b}_A)^{-1}\tilde{B}_a$ and the eigenvalues of $(\tilde{b}_A)' \otimes F_A$ are inside the unit circle. The relevant eigenvalues are α and $\frac{2\beta}{\sqrt{4\kappa\sigma(1-\beta\phi_\pi^a)+(\beta\sigma\phi_x^a+\beta+\kappa\sigma-1)^2}+\beta\sigma\phi_x^a+\beta+\kappa\sigma+1}$, which are inside the unit circle given $\phi_\pi^a < \frac{\kappa+(\beta-1)\phi_x^a}{\kappa}$ and because $|\alpha| < 1$ by construction. Thus, by Proposition 2 there is no Forward Guidance Puzzle.

The second case is when $q = \pi$. Here $\bar{b}_{T^*}$ is given by

$$\bar{b}_{T^*} = \begin{pmatrix} 0 & 0 & \frac{\beta\psi_{T^*}-1}{\alpha\kappa} & \frac{1-\beta\psi_{T^*}}{\kappa} \\ 0 & 0 & -\frac{1}{\alpha} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\psi_{T^*}}{\alpha} & \psi_{T^*} \end{pmatrix}$$

where $\psi_{T^*} = \frac{-\sqrt{(\beta\sigma\phi_x^*+\beta+\kappa\sigma-1)^2-4\kappa\sigma(\beta\phi_\pi^*-1)}+\beta\sigma\phi_x^*+\beta+\kappa\sigma+1}{2\beta}$. As before, one can verify through direct computation that the recursion is

$$\tilde{b}_j = \begin{pmatrix} 0 & 0 & \frac{\beta\psi_j-1}{\alpha\kappa} & \frac{1-\beta\psi_j}{\kappa} \\ 0 & 0 & -\frac{1}{\alpha} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\psi_j}{\alpha} & \psi_j \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\beta h(\psi_{j-1})-1}{\alpha\kappa} & \frac{1-\beta h(\psi_{j-1})}{\kappa} \\ 0 & 0 & -\frac{1}{\alpha} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{h(\psi_{j-1})}{\alpha} & h(\psi_{j-1}) \end{pmatrix}$$

given $\tilde{b}_0 = \bar{b}_{T^*}$ where $\psi_j = h(\psi_{j-1}) = \frac{\kappa\sigma\phi_\pi^a+\sigma\phi_x^a+1}{\beta\sigma\phi_x^a+\beta+\kappa\sigma-\beta\psi_{j-1}+1}$.

Once again $h(\psi)$ is quadratic and there are two roots of $h(\psi) - \psi = 0$, $\psi_A < 1 < \psi_H$. Define $\psi_U = (\beta\sigma\phi_x^a + \beta + \kappa\sigma + 1)/\beta > 1$. Then, $h'(\psi) > 0$ for all $\psi < \psi_U$, $lim_{\psi\to-\infty} h(\psi) = 0$, which implies $h(\psi) > \psi$ for all $\psi < 0$. Further, $h(0) > 0$, $h(1) < 1$, and $h(\psi) > \psi$ as $\psi \to \psi_U$ if and only if $\phi_\pi^a < \frac{\kappa+(\beta-1)\phi_x^a}{\kappa}$. Thus, $0 < h'(\psi_A) < 1 < h'(\psi_H)$. Therefore, given since $-1 < \psi_0 = \psi_{T^*} < 1$ we have $\psi_j \to \psi_A = \frac{-\sqrt{4\kappa\sigma(1-\beta\phi_\pi^a)+(\beta\sigma\phi_x^a+\beta+\kappa\sigma-1)^2}+\beta\sigma\phi_x^a+\beta+\kappa\sigma+1}{2\beta}$.

The IE-stability condition for the corresponding $\bar{\Phi}(\theta_{T^a})$ is given in the text and labeled as

$\bar{b}^2$ and satisfies the necessary condition.

Mathematica notebooks that provide direct computations are available in the online appendix. □

Proof of Proposition 4 We first prove the following Lemma. The proof of the Lemma closely follows the proof of Proposition 1 of McClung (2020).

Lemma 1. The IE-stability condition in Theorem 2 is the local asymptotic stability condition for a fixed point of (41), (42), and (43) for $\xi_t = 1, \ldots, S$.

Proof: We define $\bar{z}_j = (\bar{z}_j(1), \ldots, \bar{z}_j(S))$ for $z = a, b, c$, and $\Xi_a(\xi_t, \bar{b}_j)$, and the model's structural matrices (e.g. $A_a(\xi_t)$, $B_a(\xi_t)$) as in the main text. We can express (41), (42), and (43) for $\xi_t = 1, \ldots, S$ as $\Phi_j = T(\Phi_{j-1})$ where $\Phi_j = (\bar{a}_j, \bar{b}_j, \bar{c}_j)$. Let $\bar{\Phi}$ denote a fixed point of the T-map (41), (42), and (43) for $\xi_t = 1, \ldots, S$: $\bar{\Phi} = T(\bar{\Phi})$. If T is continuously differentiable in some neighborhood of $\bar{\Phi}$, then let $DT(\bar{\Phi})$ denote the Jacobian matrix of first derivatives of T evaluated at $\Phi = \bar{\Phi}$. If all eigenvalues of $DT(\bar{\Phi})$ are less than one in modulus, then $\bar{\Phi}$ is locally asymptotically stable (i.e. $\Phi_N = T(\Phi_{N-1}) \to \bar{\Phi}$ as $N \to \infty$ given Φ_0 in an appropriate neighborhood of $\bar{\Phi}$; see Chapter 5 and Proposition 5.2 of Evans and Honkapohja (2001) for additional information). We proceed by direct computation of $DT(\bar{\Phi})$.

To compute $DT(\bar{\Phi})$, note that $T(\Phi) = (T_a(\bar{a}, \bar{b}), T_b(\bar{b}), T_c(\bar{b}, \bar{c}))$. The system $T_b(\bar{b})$ decouples from the remaining T-map equations (i.e. the evolution of $\bar{b}_j$ does not depend on $\bar{a}_j$, $\bar{c}_j$) allowing for separate computation of $DT_b(\bar{b}) = \frac{\partial T_b(\bar{b})}{\partial \bar{b}}$ evaluated at $\bar{\Phi}$. To solve for $DT_b(\bar{b})$, we linearize $T_b(\bar{b})$ at the fixed point $\bar{\Phi}$, vectorize the linearized system and use the following identification rule to identify DT_b: if $vec(dT_b) = Qvec(d\bar{b})$ then $Q = DT_b$, where $d\bar{b} = (d\bar{b}(1), d\bar{b}(2), \ldots, d\bar{b}(S))$ and dT_b is the linearized equations. Noting that $d(G(X)^{-1}) = -G(X)^{-1}(dG)G(X)^{-1}$, we have

$$dT_B = \begin{pmatrix} (\Xi_a(1,\bar{b})^{-1}(\sum_{j=1}^S p_{1j} B_a(1) d\bar{b}(j))\Xi_a(1,\bar{b})^{-1}A_a(1))' \\ (\Xi_a(2,\bar{b})^{-1}(\sum_{j=1}^S p_{2j} B_a(2) d\bar{b}(j))\Xi_a(2,\bar{b})^{-1}A_a(2))' \\ \vdots \\ (\Xi_a(S,\bar{b})^{-1}(\sum_{j=1}^S p_{Sj} B_a(S) d\bar{b}(j))\Xi_a(S,\bar{b})^{-1}A_a(S))' \end{pmatrix}'$$

Use the rule: $vec(ABC) = C' \otimes Avec(B)$, the identification rule, and the fact that $\Xi_a(i,\bar{b})^{-1}A_a(i) =$

$\bar{b}(i)$ to obtain:

$$\begin{aligned} DT_b(\bar{b}) &= \begin{pmatrix} p_{11}\bar{b}(1)' \otimes (\Xi_a(1,\bar{b})^{-1}B_a(1)) & \cdots & p_{1S}\bar{b}(1)' \otimes (\Xi_a(1,\bar{b})^{-1}B_a(1)) \\ \vdots & \ddots & \vdots \\ p_{S1}\bar{b}(S)' \otimes (\Xi_a(S,\bar{b})^{-1}B_a(S)) & \cdots & p_{SS}\bar{b}(S)' \otimes (\Xi_a(S,\bar{b})^{-1}B_a(S)) \end{pmatrix} \\ &\equiv \left(\oplus_{k=1}^{S} \bar{b}(k)' \otimes \left(I - B_a(k) \sum_{h=1}^{S} p_{kh}\bar{b}(h) \right)^{-1} B_a(k) \right) (P \otimes I_{n^2}) \end{aligned}$$

Now turn to the equation for $\bar{a}$. It is helpful to rearrange $\bar{a}$ as $\tilde{a} = (\bar{a}(1)', \bar{a}(2)', \ldots, \bar{a}(S)')'$. This allows us to express $T_a(\bar{a}, \bar{b})$ as:

$$\begin{aligned} T_a(\bar{a},\bar{b}) &= \begin{pmatrix} p_{11}\Xi_a(1,\bar{b})^{-1}B_a(1) & \cdots & p_{1S}\Xi_a(1,\bar{b})^{-1}B_a(1) \\ \vdots & \ddots & \vdots \\ p_{S1}\Xi_a(S,\bar{b})^{-1}B_a(S) & \cdots & p_{SS}\Xi_a(S,\bar{b})^{-1}B_a(S) \end{pmatrix} \tilde{a} + \begin{pmatrix} \Xi_a(1,\bar{b})^{-1}\Gamma_a(1) \\ \vdots \\ \Xi_a(S,\bar{b})^{-1}\Gamma_a(S) \end{pmatrix} \\ &= \left(\oplus_{k=1}^{S} \left(I - B_a(k) \sum_{h=1}^{S} p_{kh}\bar{b}(h) \right)^{-1} B_a(k) \right) (P \otimes I_n)\, \tilde{a} + \begin{pmatrix} \Xi_a(1,\bar{b})^{-1}\Gamma_a(1) \\ \vdots \\ \Xi_a(S,\bar{b})^{-1}\Gamma_a(S) \end{pmatrix} \end{aligned}$$

Using the same methods as before we obtain:

$$DT_a(\bar{a},\bar{b}) = \left(\oplus_{k=1}^{S} \left(I - B_a(k) \sum_{h=1}^{S} p_{kh}\bar{b}(h) \right)^{-1} B_a(k) \right) (P \otimes I_n)$$

Finally, we consider the equation for $\bar{c}$:

$$T_c(\bar{b},\bar{c}) = \begin{pmatrix} (\Xi_a(1,\bar{b})^{-1}(\sum_{j=1}^{S} p_{1j}B_a(1)\bar{c}(j)\rho_a(j) + D_a(1)))' \\ (\Xi_a(2,\bar{b})^{-1}(\sum_{j=1}^{S} p_{2j}B_a(2)\bar{c}(j)\rho_a(j) + D_a(2)))' \\ \vdots \\ (\Xi_a(S,\bar{b})^{-1}(\sum_{j=1}^{S} p_{Sj}B_a(S)\bar{c}(j)\rho_a(j) + D_a(S)))' \end{pmatrix}'$$

Using the same methods as before we obtain:

$$DT_c(\bar{b},\bar{c}) = \sum_{k=1}^{S}\left(e_k \otimes \begin{pmatrix} p_{k1}\rho_a(1)' & \dots & p_{kS}\rho_a(S)' \end{pmatrix} \otimes \left(\left(I - B_a(k)\sum_{h=1}^{S} p_{kh}\bar{b}(h)\right)^{-1} B_a(k)\right)\right)$$

where e_k is a $S \times 1$ vector with 1 in its k-th entry and zeros elsewhere. It is straightforward to show that the eigenvalues of $DT(\bar{\Phi})$ are the eigenvalues of $DT_a(\bar{a},\bar{b})$, $DT_b(\bar{b})$, and $DT_c(\bar{b},\bar{c})$. Hence, the local stability conditions associated to a fixed point of (41)-(43) are the IE-stability conditions reported in Theorem 2. □.

Proof of Proposition 4: We establish boundedness as $\Delta_p \to \infty$. Recall that y_{ss}, the steady state of the model when $t < T^a$, does not depend on ξ_t. Therefore, we can write the impact of an FGA $\{\theta_i\}_{i=T^a,T^*}$ using equations (41), (42), and (43) as:

$$\begin{aligned} |y_{ss} - \mathbb{E}\left[y_{T^a}\right]| &= |y_{ss} - \mathbb{E}\left[\bar{a}_j(\xi_t) + \bar{c}_j(\xi_t)\omega_{T^a} + \bar{b}_j(\xi_t)y_{T^a-1}\right]| \\ &= |y_{ss} - \sum_{i=1}^{S}\bar{\pi}_i\left[\bar{a}_j(i) + \bar{b}_j(i)y_{ss}\right]| \\ &= \left|(I - \sum_{i=1}^{S}\bar{\pi}_i\bar{b}_j(i))y_{ss} - \sum_{i=1}^{S}\bar{\pi}_i\bar{a}_j(i)\right|, \end{aligned}$$

where $\bar{\pi}_i$ is the marginal density of Markov state i and $|\cdot|$ is any p-norm. By the triangle inequality, it follows that

$$\left|(I - \sum_{i=1}^{S}\bar{\pi}_i\bar{b}_j(i))y_{ss}\right| + \left|\sum_{i=1}^{S}\bar{\pi}_i\bar{a}_j(i)\right| \geq \left|(I - \sum_{i=1}^{S}\bar{\pi}_i\bar{b}_j(i))y_{ss} - \sum_{i=1}^{S}\bar{\pi}_i\bar{a}_j(i)\right|.$$

If all three conditions are satisfied, then the recursion given by equations (41) - (43) are well defined with initial condition $\bar{\Phi}_0(\theta_{T^*})$. By Lemma 1, $\Phi_{\Delta_p} \to \bar{\Phi}(\theta_{T^a})$ as $\Delta_p \to \infty$. Therefore, in the limit as Δ_p goes to infinity the impact of the FGA can be no larger (in magnitude) than $\left|(I - \sum_{i=1}^{S}\bar{\pi}_i\bar{b}(\theta_{T^a},i))y_{ss}\right| + \left|\sum_{i=1}^{S}\bar{\pi}_i\bar{a}(\theta_{T^a},i)\right|$. □

A2 THE FISHER MODEL

Sections 2 and 3 use a Fisher model of inflation determination. This appendix includes more discussion of the equilibrium properties of the model when there is active fiscal policy.

The Fisher model is an endowment economy (see Leeper and Leith, 2016 for the micro-foundations of the model). Inflation is determined in the economy by combining the household's Euler equation with rules for fiscal and monetary policy, which gives rise to the following linearized system of equations:

$$\begin{aligned} i_t &= \phi \pi_t + \bar{i} \\ i_t &= \mathbb{E}_t \pi_{t+1} \\ b_t &= \delta b_{t-1} + i_t - \beta^{-1} \pi_t \end{aligned}$$

For the simple example in section 3, we assume that $0 < \phi = \phi^a < 1$, $\delta = \delta^a > 1$ for $T^a \leq t < T^*$, $\bar{i} \neq 0$ in $t = T^* - 1$ and otherwise $\bar{i} = 0$. The system for $T^a \leq t < T^*$ can be expressed as:

$$y_t = \Gamma_a + A_a y_{t-1} + B_a \mathbb{E}_t y_{t+1}$$

where $y = (\pi, b)'$ and

$$B_a = \begin{pmatrix} \frac{1}{\phi^a} & 0 \\ 1 - \frac{1}{\beta \phi^a} & 0 \end{pmatrix} \qquad A_a = \begin{pmatrix} 0 & 0 \\ 0 & \delta^a \end{pmatrix}$$

and $\Gamma_a = (-\frac{\bar{i}}{\phi^a}, \frac{\bar{i}}{\beta \phi^a})'$ if $t = T^* - 1$, otherwise $\Gamma_a = 0_{2\times 1}$. Because $\bar{i} = 0$ when $t = T^a$, the θ_{T^a} regime admits two MSV solutions of the form: $y_t = \bar{b}(\theta_{T^a}) y_{t-1}$, where, $\bar{a}(\theta_{T^a}) = 0_{2\times 1}$ and

$$\bar{b}(\theta_{T^a}) = \begin{pmatrix} 0 & \frac{\phi^a - \delta^a}{\phi^a - \beta^{-1}} \\ 0 & \phi^a \end{pmatrix} \text{ and } \tilde{b}(\theta_{T^a}) = \begin{pmatrix} 0 & 0 \\ 0 & \delta^a \end{pmatrix}.$$

One can verify through explicit computation that the largest eigenvalues of the first solution for $DT_a(\bar{a}, \bar{b})$ and $DT_b(\bar{b})$ are $1/\delta^a < 1$ and $\phi^a/\delta^a < 1$, respectively. Hence, $\bar{\phi}(\theta_{T^a}) = (0_{2\times 1}, \bar{b}(\theta_{T^a}))$

is IE-stable.[A1] Now we discuss the implications of two alternative assumptions about the θ_{T^*} regime.

Case. Suppose $\phi = \phi^a$, $\delta = \delta^a$ for $t \geq T^*$. Then the unique dynamically stable MSV solution for $t \geq T^*$ is given by[A2]

$$y_t = \begin{pmatrix} 0 & \frac{\phi^a - \delta^a}{\phi^a - \beta^{-1}} \\ 0 & \phi^a \end{pmatrix} y_{t-1} = \bar{b}(\theta_{T^*}) y_{t-1} = \bar{b}(\theta_{T^a}) y_{t-1}$$

Thus the unique stable terminal solution is given by $\Phi_0(\theta_{T^*}) = (0_{2\times1}, \bar{b}(\theta_{T^*}))$. We now obtain the FGA solution. We first obtain the solution for $\bar{b}_j$. Because $\bar{b}(\theta_{T^*}) = (I - B_a \bar{b}(\theta_{T^*}))^{-1} A_a = \bar{b}(\theta_{T^a})$, we have $\bar{b}_1 = (I - B_a \bar{b}(\theta_{T^*}))^{-1} A_a = \bar{b}(\theta_{T^a})$ which implies $\bar{b}_j = \bar{b}(\theta_{T^a})$ for all $j \geq 0$. Now consider the recursion for $\bar{a}_j$. Since $\bar{b}_j = \bar{b}(\theta_{T^*})$ for all j we have:

$$\begin{aligned} \bar{a}_j &= (I - B_a \bar{b}(\theta_{T^*}))^{-1} B_a \bar{a}_{j-1} + (I - B_a \bar{b}(\theta_{T^*}))^{-1} \Gamma_a \\ &= DT_a(\bar{a}, \bar{b}) \bar{a}_{t-1} + (I - B_a \bar{b}(\theta_{T^*}))^{-1} \Gamma_a \end{aligned}$$

As shown above, the eigenvalues of $DT_a(\bar{a}, \bar{b})$ are strictly inside the unit circle and $\Gamma_a = 0_{2\times1}$ for all $j > 1$. Therefore $\bar{a}_j \to 0_{2\times1}$ as $j \to \infty$. This proves that $\Phi_j \to \bar{\Phi}(\theta_{T^a})$ as $j \to \infty$ given $\Phi_0 = \Phi_0(\theta_{T^*})$. The forward guidance puzzle is absent.

Case. Now suppose $\phi = \phi^* > 1$ and $\delta = \delta^* < 1$ for $t \geq T^*$. Then the unique dynamically stable MSV solution for $t \geq T^*$ is given by[A3]

$$y_t = \begin{pmatrix} 0 & 0 \\ 0 & \delta^* \end{pmatrix} y_{t-1} = \tilde{b}(\theta_{T^*}) y_{t-1}$$

Thus the unique stable terminal solution is given by $\Phi_0(\theta_{T^*}) = (0_{2\times1}, \tilde{b}(\theta_{T^*}))$. We now obtain

[A1] The θ_{T^a} regime admits two MSV solutions. The second is given by $\pi_t = 0$, and $b_t = \delta^a b_{t-1}$. It is easy to show that the IE-stability eigenvalue in this case is $1/\phi^a > 1$, and therefore this equilibrium is not IE-stable.

[A2] By Proposition 1, this MSV solution is the unique dynamically stable rational expectations solution. Note that the θ_{T^*} regime admits two MSV solutions. The second is given by $\pi_t = 0$, and $b_t = \delta^a b_{t-1}$. Because $\delta^a > 1$ the second MSV solution is not a dynamically stable equilibrium. Further note that for simplicity we assume $\phi = \phi^a$, $\delta = \delta^a$ for $t \geq T^*$, but qualitatively similar results emerge if $\delta > 1 > \phi > 0$ but $\phi \neq \phi^a$, $\delta \neq \delta^a$ for $t \geq T^*$.

[A3] The θ_{T^*} regime admits two MSV solutions. The second is given by $\pi_t = \frac{\phi^* - \delta^*}{\phi^* - \beta^{-1}} b_{t-1}$, and $b_t = \phi^* b_{t-1}$. Because $\phi^* > 1$ the second MSV solution is not a dynamically stable equilibrium.

the FGA solution. We first obtain the solution for $\bar{b}_j$. Starting from $j = 1$:

$$\bar{b}_1 = (I - B_a \tilde{b}(\theta_{T^*}))^{-1} A_a = \begin{pmatrix} 0 & 0 \\ 0 & \delta_a \end{pmatrix} = \tilde{b}(\theta_{T^*})$$

Iterating forward, we see that $\bar{b}_j = \tilde{b}(\theta_{T^*})$ for $j \geq 0$. This shows that Φ_j does not converge to $\bar{\Phi}(\theta_{T^a})$; condition (3) of Proposition 2 fails. Moreover, one can easily verify that $B_a \bar{b}_j = 0_{2\times 2}$ for $j \geq 0$ and therefore $\bar{a}_j$ evolves according to

$$\bar{a}_j \;\; = \;\; B_a \bar{a}_{j-1} + \Gamma_a = B_a^{j-1} \Gamma_a = \begin{pmatrix} -\frac{\bar{i}}{(\phi^a)^j} \\ \frac{\bar{i}(1-\beta\phi^a)}{(\phi^a)^j} \end{pmatrix}$$

where the second and third equality follow from the fact that $\Gamma_a = 0$ for $j > 1$ and $a_0 = 0_{2\times 1}$. Notice that $\pi_{T^a} = -\frac{\bar{i}}{(\phi^a)^{\Delta_p}} \rightarrow +/-\infty$ as $\Delta_p = T^a - T^* \rightarrow \infty$. We obtain the same solution as in the section 2 example under RE and the puzzle emerges.[A4]

As emphasized in the text, $\delta > 1$ is a consequence of fiscal irresponsibility. If $\delta > 1$ in all periods, then debt-stabilizing inflation is necessary for a stable equilibrium to exist. This debt-stabilizing role for inflation rules out unbounded unanticipated changes in inflation, e.g. following a forward guidance announcement, and this solves the puzzle (as found by Cochrane (2017)). However, this is not the case if the fiscal authority is expected to stabilize the debt stock in the future ($\delta < 1$ for $t \geq T^*$). Agents understand that debt stability comes from fiscal revenues in the long run ($t \geq T^*$) and this permits a unique equilibrium in which inflation plays no systematic debt-stabilizing role in any period. It is the standard "Ricardian" solution considered in our section 2 example and by most other papers in the literature. Fiscal sustainability considerations do not pin down inflation in this equilibrium, and this permits unbounded changes in inflation at the time of announcement. The puzzle emerges.

We can also directly appeal to the mathematical structure of the system to characterize why the puzzle arises when $\delta < 1$ for $t \geq T^*$. The model admits a "block-recursive" structure; the equation for inflation does not depend directly on lagged, current or expected debt, and hence equilibrium inflation will not depend on debt in the T^a regime unless it also depends

[A4] Note that here we assume the shock occurs at $t = T^* - 1$ whereas the shock is assumed to occur at T^* in the section 2 example. The results are not sensitive to the exact timing of the shock.

on debt in the T^* regime. Cho (2021) describes how the block-recursive structure of the same model *without* an anticipated structural change makes it difficult to solve the model forward.[A5] Specifically, in the case $\delta > 1$ and $\phi < 1$, one cannot obtain the unique dynamically stable MSV solution by solving the model forward (i.e. by iterating on the T-map; see Cho and Moreno (2011) and Cho (2016)) unless the solution recursion is modified to ensure that expected inflation depends on debt in the recursion. In contrast, our model is a piecewise linear model involving an anticipated structural change, and importantly, we obtain a unique rational expectations equilibrium in both cases described above. The block-recursive structure is an important feature of our model that identifies the assumptions needed in the announcement and terminal regimes to ensure that the puzzle is absent in the resulting unique equilibrium.

A3 Simple sunspot example

Consider the following bivariate system of expectational difference equations

$$y_t = \frac{1}{\phi_y}\mathbb{E}_t y_{t+1} + \frac{1}{\phi_x}\mathbb{E}_t x_{t+1} + \epsilon_t \tag{A1}$$

$$x_t = \frac{1}{\phi_x}\mathbb{E}_t x_{t+1}, \tag{A2}$$

where $\phi_y \neq \phi_x$. To capture both determinate and indeterminate FGA solutions, we study the BN augmented system. Specifically, and without loss of generality, we assume that expectations of x_t may be driven by a sunspot process v_t. In order to account for this possibility, we append

$$s_t = \frac{1}{\alpha}s_{t-1} - v_t + x_t - \mathbb{E}_{t-1}x_t, \tag{A3}$$

to the system of equations. By choosing the value of α and ϕ_x or ϕ_y appropriately, we can move between the determinate and indeterminate solutions of the model.

The system is written compactly as

$$\tilde{A}_0 z_t = \tilde{A}_1 z_{t-1} + \tilde{B}_0 \mathbb{E}_t z_{t+1} + \tilde{D}_0 \varepsilon_t,$$

[A5] Cho (2021)'s insights apply to both the model with fixed parameters, δ and ϕ, and to models that feature Markov-switching in those parameters (e.g. see section 6 of this paper).

where $z_t = (y_t, x_t, s_t, \mathbb{E}_t x_{t+1})'$, $\varepsilon_t = (\epsilon_t, v_t)'$,

$$\tilde{A}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; \tilde{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; \tilde{B}_0 = \begin{pmatrix} \phi_y^{-1} & \phi_x^{-1} & 0 & 0 \\ 0 & \phi_x^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; \tilde{D}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

RE solutions to the above take the form of

$$z_t = a + bz_{t-1} + c\varepsilon_t$$

and must satisfy the following the conditions: $a = (I - \tilde{B}b)^{-1}\tilde{B}a$, $b = (I - \tilde{B}b)^{-1}\tilde{A}$, and $c = (I - \tilde{B}b)^{-1}\tilde{D}$, where $\tilde{A} = \tilde{A}_0^{-1}\tilde{A}_1$, $\tilde{B} = \tilde{A}_0^{-1}\tilde{B}_0$, and $\tilde{D} = \tilde{A}_0^{-1}\tilde{D}_0$. The quadratic in b has closed form solutions, which define the RE solutions:

$$\bar{b}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \bar{b}^2 = \begin{pmatrix} 0 & 0 & -\frac{\phi_y}{\alpha(\phi_y - \phi_x)} & \frac{\phi_y}{\phi_y - \phi_x} \\ 0 & 0 & -\frac{1}{\alpha} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\phi_x}{\alpha} & \phi_x \end{pmatrix}$$

The first solution, $\bar{b}^1$, is the unique RE solution when the model is determinate, which implies $y_t = \epsilon_t$ and $x_t = 0$. The s_t variable is an exogenous process that does not affect y_t or x_t in this case. The second solution, $\bar{b}^2$, is an indeterminate solution that permits coordination on the sunspot v_t. The solution in this case is

$$y_t = \frac{\phi_y}{\phi_y - \phi_x} x_t + \epsilon_t \tag{A4}$$

$$x_t = \phi_x x_{t-1} + v_t. \tag{A5}$$

A3.1 Example of IE-stability analysis

To analyse the Forward Guidance properties of the model, we first calculate the IE-stability conditions for each equilibria using

$$DT_b(\bar{b}) = \left[(I - \tilde{B}\bar{b})^{-1}\tilde{A}\right]' \otimes \left[(I - \tilde{B}\bar{b})^{-1}\tilde{B}\right]$$

and

$$DT_a(\bar{a}, \bar{b}) = (I - \tilde{B}\bar{b})^{-1}\tilde{B}.$$

The eigenvalues for the two solutions are

$$Eig(DT_b(\bar{b}^1)) = \left\{0,0,0,0,0,0,0,0,0,0,0,0,0,0, \frac{1}{\alpha\phi_x}, \frac{1}{\alpha\phi_y}\right\} \text{ , } Eig(DT_a(\bar{a}^1, \bar{b}^1)) = \left\{0,0, \frac{1}{\phi_x}, \frac{1}{\phi_y}\right\}$$

$$Eig(DT_b(\bar{b}^2)) = \left\{0,0,0,0,0,0,0,0,0,0,0,0,0,0, \alpha\phi_x, \frac{\phi_x}{\phi_y}\right\} \text{ , } Eig(DT_a(\bar{a}^2, \bar{b}^2)) = \left\{0,0, \alpha, \frac{1}{\phi_y}\right\}.$$

When $\alpha > 1$, the determinate θ_{T^a} solution, $\bar{b}^1$, is selected when using standard solution techniques. IE-stability requires that $|\phi_x| > 1$ and $|\phi_y| > 1$, which coincide exactly with the conditions for determinacy.

When $\alpha < 1$ and $|\phi_x| < 1$, standard solution techniques select the sunspot solution. The IE-Stability conditions in this case reduces to $|\phi_y| > 1$. When this solution is selected it is possible that the forward guidance puzzle is not present. The indeterminacy implied by ϕ_x no longer matters for the IE-stability condition.

This illustrates the confusion discussed in the introduction, which ascribes indeterminacy to the forward guidance puzzle. The determinacy or indeterminacy of the θ_{T^a} regime just happen to coincide with IE-stability conditions in the first case. But in the second case, IE-stability and determinacy do not imply the same conditions.

To see that the IE-stability condition is the relevant condition for diagnosing a forward guidance puzzle, it is useful to work through an example. Consider an FGA where a temporary change to x_t is announced at date $t = T^a$ but which occurs at time $t = T^*$.[A6] To induce

[A6] Whether the change occurs in y_t or x_t does not affect the conclusion of this analysis.

indeterminacy in the θ_{T^*} regime, we assume that $|\phi_x| < 1$ such that

$$x_t = \phi_x x_{t-1} + v_t$$

for $t > T^*$. The sunspot solution for $t > T^*$ follows equations (A4) and (A5). The structural equations at the $t = T^*$ are

$$\begin{aligned} y_t &= \frac{1}{\phi_y}\mathbb{E}_t y_{t+1} + \frac{1}{\phi_x}\mathbb{E}_t x_{t+1} + \epsilon_t \\ x_t &= \gamma + \frac{1}{\phi_x}\mathbb{E}_t x_{t+1}, \end{aligned}$$

where γ is the anticipated change. The backward recursion starts by constructing the expectations that prevail in time period $t = T^*$ given the above equations and the sunspot terminal solution:

$$\mathbb{E}_t x_{T^*+1} = \phi_x(x_{T^*} - \gamma) \text{ and } \mathbb{E}_t y_{T^*+1} = \frac{\phi_y}{\phi_y - \phi_x}\phi_x(x_{T^*} - \gamma).$$

Then, substituting these beliefs into the structural equations, we recover

$$\begin{aligned} \mathbb{E}_{T^a} y_{T^*} &= \frac{1}{\phi_y}\left(\frac{\phi_y}{\phi_y - \phi_x}\phi_x(x_{T^*} - \gamma)\right) + \frac{1}{\phi_x}\left(\phi_x(x_{T^*} - \gamma)\right) \\ \mathbb{E}_{T^a} x_{T^*} &= \gamma + \frac{1}{\phi_x}\phi_x(x_{T^*} - \gamma) = \phi_x x_{T^*-1}, \end{aligned}$$

where the last equality is obtained because we assume that agents coordinate on a sunspot. Finally, simplifying and working our way back through time we can recover the following solution path

$$\begin{aligned} \mathbb{E}_{T^a} y_{T^*-1} &= \frac{1}{\phi_y}\left(\frac{\phi_y}{\phi_y - \phi_x}\right)(-\gamma) + \frac{\phi_y}{\phi_y - \phi_x}x_{T*-1} \\ \mathbb{E}_{T^a} x_{T^*-1} &= \phi_x x_{T^*-2} \\ &\ldots \\ y_{T^a} &= \left(\frac{1}{\phi_y}\right)^{T^*-T^a}\left(\frac{\phi_y}{\phi_y - \phi_x}\right)(-\gamma) + \frac{\phi_y}{\phi_y - \phi_x}x_{Ta} \\ x_{T^a} &= \phi_x x_{T^a-1} + \epsilon_{T^a}, \end{aligned}$$

where the last period is the contemporaneous effect of the announcement. The impact of the FGA is given by

$$\left(\frac{1}{\phi_y}\right)^{\Delta_p}\left(\frac{\phi_y}{\phi_y-\phi_x}\right)\gamma.$$

The impact is bounded as $\Delta_p = T^* - T^a \to \infty$ when $|\phi_y| > 1$. Exactly as predicted by IE-stability. The indeterminacy caused by ϕ_x no longer plays a role. The reason that this occurs is because we assume that the agents coordinate on the sunspot in the terminal regime. The agents' beliefs about the evolution of x_t from that point on are pinned down by the sunspot process. They are "predetermined" by last period's realizations in $t = T^a$. They no longer respond at all on announcement of the policy. From the standpoint of calculating the expectations of y_t, the evolution of x_t is exogenously determined.

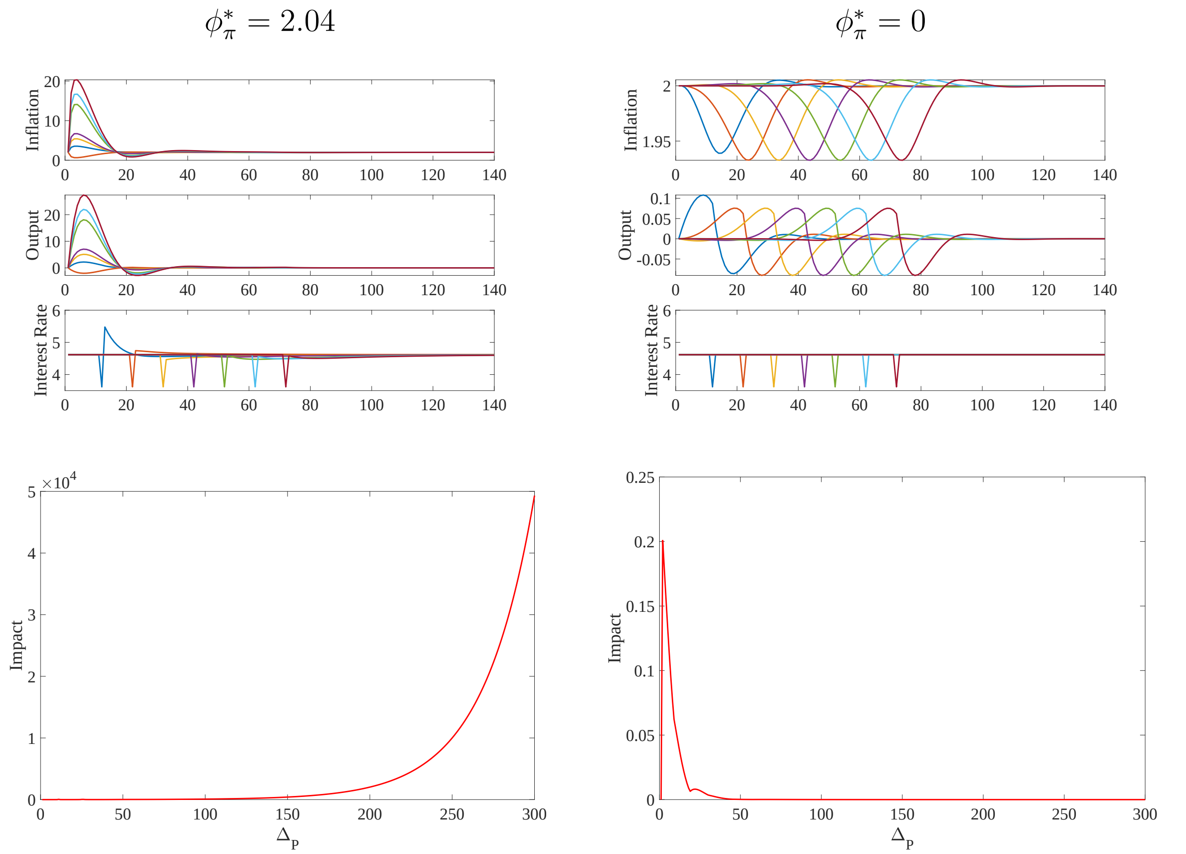
A4 Medium Scale model robustness

Figure A6 shows the simulated response of inflation, output, and the interest rate to anticipated 100-basis point decreases in the interest rate at different horizons under an interest rate peg in the θ_{T^a} regime combined with either a determinate monetary policy announced in the θ_{T^*} regime or a continuation of the interest rate peg in a sunspot equilibrium. The value of the reaction coefficient for inflation in the determinate case is set at the posterior mean reported by Smets and Wouters (2007) ($\phi^*_\pi = 2.04$), while the other parameters of the policy rule are set to zero. For the sunspot case, all parameters of the policy rule are set to zero. The remaining parameters of the model are set to the mean posterior values reported in Smets and Wouters (2007).

The figure shows that the assumption of a return to a determinate policy regime leads to the forward guidance puzzle. It also generates a reversal puzzle in this case. For the case where the anticipated interest rate change is in 20 quarters, inflation falls at the announcement rather than rises. This additional puzzle was also pointed out by Carlstrom et al. (2015). It occurs when the unstable eigenvalues are complex. The assumption of an indefinite interest rate peg in the sunspot equilibrium does not lead to the forward guidance puzzle.

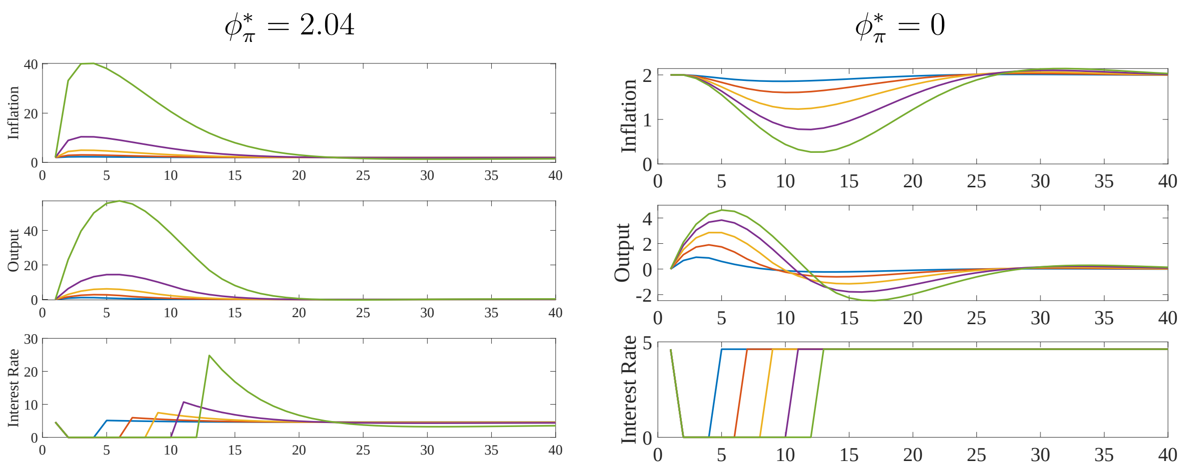
Figure A7 shows the same experiment but for a case of an actual announced zero lower bound regime. Here the interest rate is dropped from a positive steady state to zero and it is announced that it will remain at zero for 4, 6, 8, 10, or 12 quarters. The drop in the interest rate

Figure A6: Forward Guidance Announcements in the Smets and Wouter's model



Notes: The top figures shows the paths of output, inflation, and interest rates for anticipated 100-basis point monetary policy shocks with Δ_p = 10, 20, ... , 70. The bottom figures show the impact using the L_∞ norm for $\Delta_p = 1, ..., 300$.

Figure A7: Forward Guidance Announcements at the ZLB in the Smets and Wouter's model



Notes: The figure shows the paths of output, inflation, and interest rates for anticipated bind of the zero lower bound for Δ_p = 4, 6, 8, 10, and 12 quarters.

in this case is 400+ basis points, which causes a much larger output and inflation response in the determinate terminal regime case. This illustrates the type of stimulus that would be provided by making these announcements at the zero lower bound. Promises to keep interest rates at zero beyond the duration of a shock that caused the zero lower bound to bind are in effect promises of future 400+ basis point monetary policy interventions in the absence of further shocks in this model. A forward guidance announcement of this kind is highly stimulatory in the determinate case. In the indeterminate case, policy is an interest rate peg in both the zero lower bound regime and afterwards. The central bank moves the interest rate peg to zero and announces a promised duration of zero interest rates. It then moves the interest rate back to steady state and pegs the rate there. In this case, the policy increases output but results in a mild deflation. There is no forward guidance puzzle.

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