# Strongly Rational Routes to Randomness

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**Abstract**

Many behavioral models of expectation formation in macroeconomics and finance assume agents choose from a menu of predictor rules over time to construct forecasts. The menus often include a costly rational predictor that yields a correct prediction taking into account the heterogeneity of expectations. In this paper, I study the eductive justification of this rational predictor in the sense of Guesnerie (2002). I show that the condition for eductive stability is strictly weaker than in the homogeneous expectations case under commonly held assumptions. This implies that many rational routes to randomness identified by Brock and Hommes (1997) are actually strongly rational and that rational forecasting behavior is not necessarily ruled out when eductive stability in the homogeneous expectations case fails. The latter has implications for when rational behavior may occur in laboratory experiments on expectations.

**JEL Classifications**: E31; E32; D84; D83; C92

**Key Words**: Expectations; Cobweb model; Eductive Learning; Stability.

## 1 Introduction

Brock and Hommes (1997) (BH) put forward a model of evolutionary learning and rationally heterogeneous expectations (RHE) where agents choose from a menu of predictor rules of varying costs to calculate forecasts. The key idea is that people contemplate different strategies of varying degrees of sophistication, and that over time they switch strategies based on past performance and the cost of implementation. Typically, a costly

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rational predictor that yields a correct expectation taking into account the possible heterogeneity of expectations among agents is included on the menu. For example, this is the approach taken by Brock and Hommes (1998) and Brock et al. (2009) in asset pricing models; in Branch and McGough (2010) in the New Keynesian model; in Branch and McGough (2016) in a monetary search model; and in Brock et al. (2006) and Branch and McGough (2004, 2008) in the Cobweb model.

The assumption of a rational predictor, though, is a strong assumption. Hommes (2013) notes that it requires far more structural information than is required under standard Rational Expectations (RE) in the homogeneous expectations case because agents must be aware of the predictions and distribution of nonrational agents in the market.[1] In addition, Learning-to-Forecast laboratory experiments such as Hommes et al. (2005), Hommes et al. (2007), Hommes et al. (2008), Anufriev and Hommes (2012), Hommes and Lux (2013), Hüsler et al. (2013) or Agliari et al. (2016) find support for heterogeneous expectations and evolutionary learning but strongly reject the standard RE hypothesis, which has further cast doubt on the plausibility of the rational predictor. The theoretical explanations of the forecasting behaviour observed in these papers have, therefore, excluded the possibility of a rational predictor in favor of simple behavioral rules.[2] However, it is not clear that strategic rules of this nature should be completely ruled out given the finding of heterogeneous expectations and the fact that RHE predictions depart substantially from homogeneous RE. Information aside, it is also not known whether the rational predictor actually implies more rationality on the part of the agents than the standard RE assumption.

In this paper, I study the plausibility of the rational predictor using the educative stability framework of Guesnerie (1992, 2002). Specifically, I ask under what conditions the agents who choose the rational predictor can coordinate on a unique rational prediction assuming structural knowledge of the economy and the common knowledge of rationality. Predictions that are eductively stable under Guesnerie's framework are said to be *strongly rational*. I show, perhaps surprisingly, that the condition for eductive stability of the rational predictor is strictly weaker than in the homogeneous expectations case under commonly held assumptions.

The reason rational agents can coordinate in these situations is that nonrational agents bring certainty. Their actions are deterministic and common knowledge among the rational agents. This narrows the set of plausible outcomes. Furthermore, because only a fraction of agents are rational, their overall effect on the marked outcome is diminished.

[1] See page 135 of Hommes (2013).

[2] Surveys of this literature may be found in Hommes (2011) and Hommes (2013).

This again narrows the set of plausible outcomes, which facilitates coordination.

The finding is significant because it shows that strategic behavior approximating the rationality of the rational predictor is not as implausible as previously thought. Although, it is true the information barriers are high to actually obtaining a correct expectation. The individual rationality required to deduce a forecast given this information is plausible. In addition, rationally motivated forecasts are also reasonable to consider even in the absence of perfect information. If people simply posit likely values for the distribution of agents and forecasts, and consider other "rational" people to be like minded, then they could deduce a rationally motivated forecast. In fact, the more nonrational people postulated to exist, the easier it becomes to work out a unique expectation to hold. This is in contrast to the eductively unstable case, where it is impossible to narrow down the set of possible forecasts regardless of information, which argues strongly for only considering behavioral strategies.

Bao and Duffy (2016) provide some experimental support for rational considerations of this type. They study whether adaptive or eductive learning better explains forecasting behavior in the Cobweb model by exploiting the fact that conditions for stability under adaptive learning and eductive stability do not always overlap. They find evidence for both eductive and adaptive learning behavior in all cases. This includes cases where the model is eductively unstable in the homogeneous expectations case. The theoretical results presented here offer an explanation for why eductive behavior may occur in these situations given the presence of and belief of heterogeneous expectations. Indeed, the belief of heterogeneous expectations may actually encourage strategically motivated forecasts among some experimental participants.

As an application of the main result, I analyze the eductive stability of the BH model to show the existence of strongly rational routes to randomness. The BH model is particularly interesting to study here because the necessary condition identified by BH for chaotic dynamics to emerge is mutually exclusive of the condition for eductive stability obtained by Guesnerie (1992) for the homogeneous expectations case.[3]

The remainder of the paper proceeds as follows. Section 2 introduces strong rationality and establishes eductive stability of the rational predictor in a general setting. Section 3 applies strong rationality to the model of BH and provides an example of a strongly rational route to randomness. Section 4 concludes.

[3]Hommes and Wagener (2010) also considers educative learning in the BH model. However, they focus on whether the unique REE is evolutionarily stable in the eductively stable case under homogenous expectations. This application is concerned with the eductively unstable case.

## 2 Strong Rationality in a General Framework

Guesnerie (2002) puts forward a general framework to study the coordination of expectation in economic models. He focuses on models with a "muthian" structure, where a continuum of agents make simultaneous economic decisions today based on expectations of the state of the economy tomorrow.[4] He represents this economy by $T(X)$, where $X$ is $n \times 1$ vector of beliefs over the aggregate states of the economy and $T : \mathbb{R}^n \to \mathbb{R}^n$ that maps aggregate expectations to aggregate outcomes. A Rational Expectation Equilibrium (REE) of $T(X)$ is defined as

$$T(\bar{X}) = \bar{X}. \tag{1}$$

The question of interest here is under what conditions can agents coordinate on $\bar{X}$ from the perspective of a static simultaneous move game assuming only the common knowledge rationality and some bound on possible beliefs. To this aim, Guesnerie introduces the following definitions:

**Common Knowledge of Rationality (CK):** *The individual knowledge possessed by a rational agent of the structure of the economy and the knowledge that all other agents are rational and know the structure of the economy and know that all agents know that the other agents know and so on ad infinitum.*

**CK restriction:** *It is common knowledge that all plausible beliefs lie in some open set $\Omega$ such that $\bar{X} \in \Omega$.*

**Local Strong Rationality:** *An equilibrium is said to be locally strongly rational if given that it is $CK$ that $X \in \Omega(\bar{X})$, then it is $CK$ that $X = \bar{X}$ .*

Local strong rationality is akin to the game theoretic notion of rationalizability. An equilibrium is locally strong rational if it is the unique rationalizable expectation for each individual agent to hold, obtained independently, through iterated deletion of strictly dominated strategies.

In practice, this means that $T(X)$ is a contraction mapping for beliefs $X^e \in \Omega(\bar{X})$. A rational agent who chooses an arbitrary $X^e \in \Omega(\bar{X})$ can deduce using $T(X)$ that the actual market outcome must lie in some set $\Omega^0 \subset \Omega(\bar{X})$, and by repeating the exercise can

[4] The term Muthian refers to the model studied by Muth (1961) in which the Rational Expectations Hypothesis was first discussed.

deduce that $\Omega^\infty = \bar{X}$ is the only rational belief to hold. Guesnerie argues that eductive reasoning is a type of learning that occurs in the minds of agents. Agents do not know the correct expectation to hold at the beginning of this deduction but because of CK they can iterate $T(X)$ on an initial belief to arrive at $\bar{X}$. On the other hand, if eductive stability fails, then individual agents cannot deduce a unique expectation to hold. Any posited belief results in a larger set of possible outcomes. Guesnerie (2002) notes in the case of homogeneous agents and point expectations that the REE $\bar{X}$ is locally unique if

$$\det\left(I_{n\times n} - A(\bar{X})\right) \neq 0, \tag{2}$$

where $A(\bar{X}) = \partial T/\partial X|_{\bar{X}}$ and it is locally strongly rational if the spectral norm of $A(\bar{X})$ is less than 1.

Finally, it is important to stress here that the ability to conduct this mental deduction and arrive at a unique belief is completely separate from whether or not that belief is correct. The other agents after all may not be rational. But, without eductive stability the mental process cannot converge. Therefore, the use of a behavioral or rule-of-thumb forecasts make sense to consider in the eductively unstable case even when agents are fully rational.

## 2.1 Strong Rationality and Heterogeneous Agents

Evans and Guesnerie (1993) and Guesnerie (2002) expand the notion of strong rationality to include the possibility of structural heterogeneity and relax the assumption of point expectations. They propose the following assumptions and prove the following proposition:

A1: There is a continuum of agents on the unit interval, $U$, indexed by $j$ whose individual effect on the $T(X)$ for $X_j^e$ in a neighborhood of $\bar{X}$ is $A(j,\bar{X})$ such that $A(j,\bar{X}) = \partial T/\partial X_j|_{\bar{X}}$.

A2: Each agents expectation is described as a random variable $\tilde{X}(j)$, whose support is in $\Omega(\bar{X})$ with mean $X(j)$. For $\Omega(\bar{X})$ small enough, the state $X'$ of the system is, to the first order approximation is given by

$$X' - T(\bar{X}) = \int_U A(j,\bar{X})[X(j) - \bar{X}]dj.$$

**Evans and Guesnerie's Theorem:** *Assume A1 and A2, then the equilibrium* $\bar{X} \in \Omega$ *is Locally Strongly Rational if*

$$\int_U ||A(j, X)||dj < 1, \tag{3}$$

*where* $||.||$ *is the spectral norm.*

I refer the reader to Evans and Guesnerie (1993) for a formal proof of the theorem and to Guesnerie (2002) for a more in depth overview of the concept as a whole and comment here only on the intuition and two notable properties. The intuition is straightforward. If Equation (3) is satisfied, then for any initial belief $X_0^e \in \Omega(\bar{X})$ the local approximation implies that the state of system is closer to $\bar{X}$ than the initial belief. Repeated applications of this deduction converge to $\bar{X}$.

The first notable property of the theorem is that it relies on the state of the economy being additive across agents and that it only depend on the mean of agents' expectations. Guesnerie (2002) notes that these are all standard properties of first order approximations commonly studied in economics. They are also relatively reasonable assumptions to expect actual people to make when analyzing a complex environment. The second property is that Evans and Guesnerie's Theorem is general because it permits rational agents to be of heterogeneous types. Heterogeneity of types is actually rarely considered in RHE models. Agents are typically assumed to be *ex ante* identical.

## 2.2 Strong Rationality, Heterogeneous Agents, and Rationally Heterogenous Expectations

RHE as defined by BH starts with the premise that agents possess a menu of predictor rules to forecasts the endogenous states of the economy. Let the menu or predictor rules be denoted as $\{\hat{X}_i\}_{i=1}^N$, where without loss of generality $\hat{X}_1$ is the rational predictor. The remaining predictors are assumed to be deterministic and not reliant on strategic thinking. There are possibly hetergeneous types as well as heterogeneous agents in this setting. To account for this, let $\{\mathbf{w}_i\}_{i=1}^N$ be a collection of Lebesque measurable sets that partition $U$ according to each agents predictor rule choice, where the aggretate expectation under RHE is

$$X^e = \int_U \hat{X}_i d\mu = \sum_{i=1}^N \omega_i \hat{X}_i, \tag{4}$$

such that $\sum_{i=1}^{N} \omega_i = 1$.

To close the model, BH proposes that agents select predictor rules in each period using a fitness measure that depends on past $X$'s and $\hat{X}$'s, and which is subject to some intrinsic uncertainty. The intrinsic uncertainty causes all predictor rules to be used by some proportion of agents in each period. Though, the most fit rules always attract the highest proportion of users. Fitness measures typically correspond to the past observed profits of each predictor rule. The fitness rule is assumed to be CK and therefore so are the proportion of agents who choose each rule.

The agents who choose the rational predictor must work out $X_1$ such that

$$T\left(\omega_1 \bar{X}_1 + \sum_{i=2}^{N} \omega_i \hat{X}_i\right) = \bar{X}_1. \tag{5}$$

Here again the question of interest is when is $\bar{X}_1$ eductively stable. To assess this, consider a first order approximation of Equation (5) assuming A1 and A2 around $\bar{X}_1$:

$$X = T\left(\omega_1 \bar{X}_1 + \sum_{i=2}^{N} \omega_i \hat{X}_i\right) + \int_{\mathbf{w}_1} A(j, \bar{X}_1)[X(j) - \bar{X}_1]dj, \tag{6}$$

where $A(j, \bar{X}_1) = \partial T / \partial X_j|_{\bar{X}_1}$. The approximation does not depend on $\{\hat{X}_i\}_{i=2}^{n}$ except through $\bar{X}_1$ because these forecasts are predetermined and known to the selectors of the rational predictor.

**Proposition 1:** *Assume A1, A2, and $\{\hat{X}_i, \mathbf{w}_i\}_{i=2}^{N}$ is CK, then $\bar{X}_1$ is Locally Strongly Rational if*

$$\int_{\mathbf{w}_1} ||A(j, \bar{X}_1)|| dj < 1,$$

*where $||.||$ is the spectral norm.*

The proposition follows immediately from Evans and Guesnerie's Theorem.

Depending on the model, this condition may be weaker or stronger than in the homogeneous RE case because $\bar{X}$ is not in general equal to $\bar{X}_1$. Therefore, the approximate effect of a rational agent's beliefs in the two cases, $A(j, \bar{X})$ and $A(j, \bar{X}_1)$, on the state of the economy is also not equal. For example, if the effect of rational agents' beliefs on the economy are significantly larger near $\bar{X}_1$ than near $\bar{X}$, then the condition may be stricter. On the other hand, if the rational agents' effect on the state of the economy is equal to or even larger by an appropriately small amount than in the RE case, then the condition is weaker because the overall mass of rational agents is smaller, which diminishes their overall effect on the economic outcomes. In addition, the structural heterogeneity can also effect whether the condition is weaker or stronger. If the agents who have the largest effect on the economy as measured by $A(j, \bar{X}_1)$ are also the agents who select the rational predictor, then the condition may be stronger and vice versa.

A clear example of when the condition for eductive stability is weakened is the case of a linear model, which implies $A(j, \bar{X}) = A(j, \bar{X}_1)$ for all $j$.

**Corollary 1:** *If $A(j, \bar{X}) = A(j, \bar{X}_1)$ for all $j$ and $\bar{X}_1$, and $\omega_1 < 1$, then*

$$\int_{\mathbf{w}_1} ||A(j, \bar{X}_1)||dj < \int_U ||A(j, \bar{X})||dj \tag{7}$$

**Proof:** Let $\int_U ||A(j, \bar{X})||dj = \lambda$, then

$$\begin{aligned} \int_{\mathbf{w}_1} ||A(j, \bar{X}_1)||dj &< \int_U ||A(j, \bar{X})||dj \\ \omega_1 \lambda &< \lambda \\ \omega_1 &< 1 \end{aligned}$$

□

The overall results here have a nice economic intuition. In general, as the proportion of rational agents in the market decreases, the proportion of agents whose prediction are known with certainty increases. As certainty increases, coordination among the remaining rational agents becomes easier.

From the perspective of an individual agent, there is an element of taking advantage of the less sophisticated players in RHE. Knowledge of nonrational agents' expectations always narrows the set of possible market outcomes pushing the rational agents beliefs closer to the actual outcome. Therefore, there is an incentive to use this information even when eductive stability fails.

# 3 Application to the model of Brock and Hommes

BH, Guesnerie (1992), and Bao and Duffy (2016) all study the Cobweb model.[5] Therefore, it serves as a natural laboratory to illustrate the general result. The model in this section is linear so the "local" in local strong rationality is dropped.

## 3.1 The Cobweb

The basic Cobweb assumes that a continuum of identical firms produce a non-storable good with a production lag. Production decision are made at time $t$ and finished goods are sold at time $t+1$. The market demand at time $t+1$ is given by

$$D(p_{t+1}) = A - Bp_{t+1}; \; A > 0, \; B > 0. \tag{8}$$

The firms supply to the market based on their expectation of price at time $t+1$, which is denoted as $p^e_{j,t+1}$. The firm's expectation is used to produce a profit-maximizing quantity according to

$$S(p^e_{j,t+1}) = argmax_{q_{j,t}}\{p^e_{j,t+1}q_{j,t} - c(q_{j,t})\}. \tag{9}$$

The firms each face an identical cost function,

$$c(q_{j,t}) = \frac{q^2_{j,t}}{2b}, \; b > 0, \tag{10}$$

which results in a unique profit-maximizing quantity of goods to supply to the market given an expectation of price:

$$S(p^e_{j,t+1}) = bp^e_{j,t+1}. \tag{11}$$

## 3.2 RHE in the Cobweb

Assuming RHE, the firms form expectations by choosing from a menu containing two predictor rules: a rational predictor rule, $p^{1,e}_{t+1} = p_{t+1}$, which is subject to the cost $C \geq 0$, and a naive predictor, $p^{2,e}_{t+1} = p_t$, which is free. The fraction of firms that choose the rational predictor is denoted $n^1_t$ and the fraction of that choose the naive predictor is denoted $n^2_t$. Given the fractions of firms, the market clearing condition is

$$D(p_{t+1}) = n^1_t S(p^{1,e}_{t+1}) + n^2_t S(p^{2,e}_{t+1}). \tag{12}$$

[5]The notation in this section follows subsequent work on this model by Brock et al. (2006)

Following BH the firms select predictor rules by comparing the past realized profits of each rule, net the cost of implementation, without considering how contemporaneous choices of the other agents may affect their actual payoff. The past profit of each predictor is given by

$$\pi_t^i = \pi(p_t, p_t^{i,e}) = \frac{b}{2} p_t^{i,e}(2p_t - p_t^{i,e}) - C_i \tag{13}$$

for $i = 1, 2$.

The fraction of firms that choose each predictor is determined by

$$n_{i,t} = \frac{e^{\beta F(p_{t+1}^{1,e})}}{\sum_{j=1}^{2} e^{\beta F(p_{t+1}^{j,e})}}, \tag{14}$$

where $F(p_{t+1}^{1,e}) = \pi_t^1 - C$, $F(p_{t+1}^{2,e}) = \pi_t^2$, and $\beta$ is the intensity of choice parameter that governs the amount of uncertainty that exists over the best predictor rule to choose in each period. BH defines the proportion of firms that choose each rule as $m_t = n_t^1 - n_t^2$ and substitutes the predictor rules into the market clearing condition (12) to yield the following system of equations[6]

$$A - Bp_{t+1} = \frac{b}{2}(p_{t+1}(1 + m_t) + p_t(1 - m_t)) \tag{15}$$

$$m_{t+1} = Tanh[\frac{\beta}{2}(\frac{b}{2}(p_{t+1} - p_t)^2 - C)], \tag{16}$$

which BH call Adaptive Rational Equilibrium Dynamics (ARED).

BH show that interesting and complex dynamics are predicted by ARED when the relative slopes of supply demand are such that $b/B > 1$. BH show there are four distinct dynamic predictions for the model given a choice of $\beta$ and $C$ in this case:[7]

1. If $C = 0$, then there exists a globally stable steady state $\left(\frac{A}{B+b}, 0\right)$.

2. If $C > 0$ and $0 \leq \beta < \beta_1$ for some critical value $\beta_1$, then there exists a globally stable steady state $\left(\frac{A}{B+b}, Tanh\left[\frac{-\beta C}{2}\right]\right)$, while for $\beta > \beta_1$, the steady state is an unstable saddle path.

3. If $C > 0$ and $\beta_1 < \beta < \beta_2$, then there exists a locally unique and stable 2 cycle given by $\left\{\left(\tilde{p}, \frac{-B}{b}\right), \left(-\tilde{p}, \frac{-B}{b}\right)\right\}$.

4. If $C > 0$ and $\beta > \beta_2$, then chaotic processes may emerge.

[6] The hyberbolic tangent function is a convenient way to write equation (14) in the case of two predictors.

[7] A precise statement of the dynamic properties of ARED is given in the appendix in the proof of Proposition 3.

## 3.3 Strong Rationality in the Cobweb

The necessary condition of $b/B > 1$ for complicated dynamics to emerge is particularly interesting in the current context because it implies eductive instability in the homogeneous expectations case. To illustrate, consider Equation (12) with $n_1 = 1$ and written in the reduced form of Section 2:

$$T(p) = \frac{A}{B} - \frac{b}{B}p. \tag{17}$$

The relevant condition for strong rationality of the unique rational prediction is $b/B < 1$. Therefore, the condition under which the interesting dynamics emerge in ARED is mutually exclusive of the condition under which the REE is strongly rational in the homogeneous expectations case.

Strong rationality of the rational predictor in ARED is more complicated to assess. In the Cobweb model under homogeneous expectations, there is a unique rational prediction so reducing the model a one-shot game is natural. In ARED, however, the unique rational prediction may be different in each time period. Therefore, I introduce the following definition:

**Definition:** *ARED is strongly rational if the rational predictor is strongly rational in the one-shot game period by period.*

To facilitate the analysis, I remove the $t$-subscripts from all variables and denote the forecast of the rational predictor as $\bar{p}_1$ and the forecast of the naive predictor as $p_{-1}$. Following Section 2, the reduced form of the model is given by

$$T\left(\bar{p}_1\frac{(1+m)}{2} + p_{-1}\frac{(1-m)}{2}\right) = \frac{A}{B} - \frac{b}{B}\left(\bar{p}_1\frac{(1+m)}{2} + p_{-1}\frac{(1-m)}{2}\right),$$

where $p \in [0, A/B] = \Omega(\bar{p}_1)$ and $\bar{p}_1 = \frac{2A-b(1-m)p_{-1}}{2B+b(1+m)}$. Note that $\bar{p}_1$ depends on $p_{-1}$, which may change over time.

**Proposition 2:** *The rational prediction $\bar{p}_1$ is strongly rational if*

$$\frac{b}{B}\left(\frac{1+m}{2}\right) < 1. \tag{18}$$

The propositions follows directly from the application of Proposition 1 noting that $(1+m)/2 = n_1$ is the proportion of rational agents. For a positive fraction of non-rational

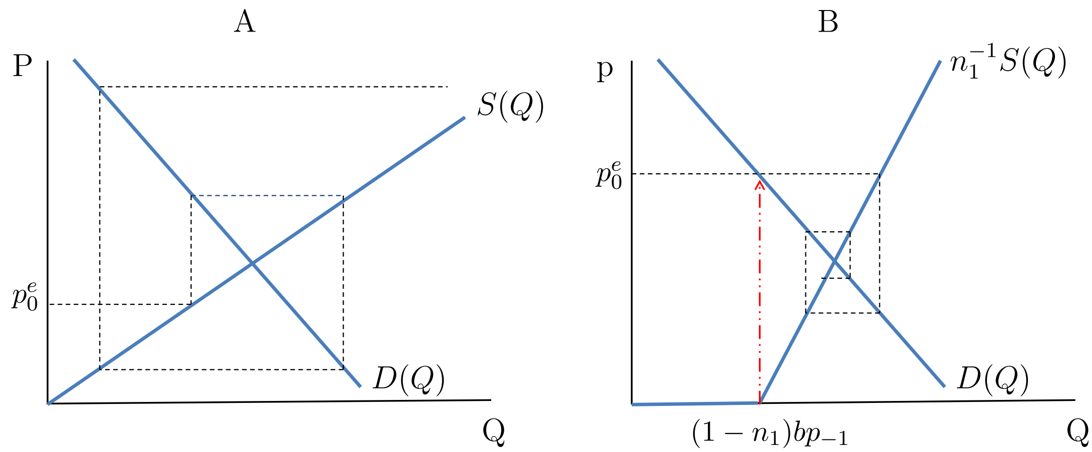


**Figure 1:** Illustration of the mental deduction in the homogeneous RE case (A) and the RHE case (B) with $n_1$ rational agents and $1 - n_1$ naive agents.

agents ($m < 1$), the condition for strong rationality is weaker than in the homogeneous expectations case.

Figure 1 illustrates the mental deduction done by the rational agents in the $n_1 = 1$ and the $n_1 < 1$ cases when $b/B > 1$. In the homogeneous RE case depicted in panel A, the agents choose an initial belief to hold, $p_0^e$, and use their knowledge of supply and demand from CK to assess the likelihood of this outcome. The supply and demand curves are such that this initial belief is not ruled out. In fact, the entire set of feasible prices is rationalizable. However, in the heterogeneous expectations case depicted in panel B, it is known with certainty that $(1 - n_1)bp_{-1}$ will be supplied to the market, which implies the price will at least be $p_0^e$. Furthermore, because the proportion of rational agents is $n_1$, the supply response for any postulated price by these agents is $n_1 bp^e$. This causes the notional supply curve to rotate leftward restoring the eductive stability of the unique rational prediction in the period so long as $n_1$ is sufficiently small.

The eductive stability properties of ARED are given by proposition 3.

**Proposition 3:** *Assume $b/B > 1$, then*

1. *when $C = 0$, the steady state $\left(\frac{A}{B+b}, 0\right)$ is not strongly rational.*
2. *when $C > 0$, $Tanh[-\beta C/2] < 2B/b - 1$, and $0 \leq \beta < \beta_1$ for some critical value $\beta_1$, the steady state $\left(\frac{A}{B+b}, \mathrm{Tanh}\left[\frac{-\beta C}{2}\right]\right)$ is strongly rational.*
3. *when $C > 0$, $1 < b/B < 3$, and $\beta_1 < \beta < \beta_2$, ARED is strongly rational, meaning each point of the period 2 orbit is strongly rational.*

The proof of proposition 3 is given in the appendix. The key to strong rationality is the positive cost of the rational predictor. Positive costs guarantee that some proportion of

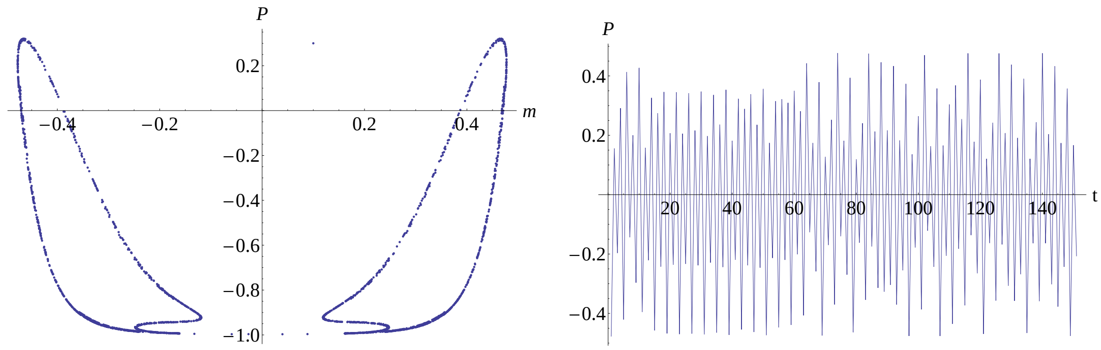


**Figure 2:** An example of a strongly rational route to randomness. The left panel depicts the chaotic attractor in $p$ and $m$ space. The point (.1, 0.3) is the initial condition of the simulation. The right panel shows a time series of the market price.

agents will select the naive predictor each period. The remaining conditions ensure that $m$ is sufficiently small such that Proposition 2 is satisfied at each point in time.

Finally, chaotic orbits also may be strongly rational. Figure 2 shows an explicit example of a strongly rational route to randomness. The parameter values for the simulation are $A = 0$, $B = 1$, $b = 1.5$, $C = 0.5$, $\beta = 13$. Note that $m_t$ is bounded on the attractor such that $m_t < 1/3$ for all $t$, which satisfies Proposition 2 for the given calibration. Therefore, there exists strongly rational routes to randomness.

## 3.4 Discussion

Now of course the predictions of ARED, even when eductively stable, require the selectors of the rational predictor to possess an immense amount of information. The point here is not that this model is an exact description of how people form expectations. The point is that, given some information about the heterogeneity of expectations, the ability to make a strategic deduction like the one capture by the rational predictor is plausible. And, more importantly, it is testable. For example, if computerized naive players are introduced into an experimental setting like the one studied by Bao and Duffy (2016) and the proportion of these players is disclosed to the actual laboratory participants, then the model makes predictions about when and how players should be able to use that information.

# 4 Conclusion

Heterogeneity of expectations and forecasting strategies are observed in both laboratory and survey forecasts.[8] Theoretical explanations of these empirical regularities usually rely on purely behavioral forecasting rules that exclude strategic considerations. This paper shows, however, that strategic considerations become more plausible by at least one metric, educative stability, when there exists heterogeneous expectations. This opens up the possibility that strategic or rational behavior may explain some behaviors observed in Learning-to-Forecast experiments and that predictions made by RHE models that include rational predictors may be more plausible than previously thought.

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[8]See Branch (2004, 2007) for evidence of heterogeneous expectations and forecasting strategies in survey forecasts.

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# 5 Proof Appendix

**Proof of Proposition 3:** Theorem 3.1 in Brock and Hommes (1997) and Theorem 1 in Brock et al. (2006) summarizes the dynamic properties of ARED given by Equations (15) and (16):

*Assume $b/B > 1$, then*

1. *When $C = 0$, the steady state $(\bar{p}, 0)$ is always globally stable.*

2. *When $C > 0$, then there exists a critical value $\beta_1$ such that for $0 \leq \beta < \beta_1$ the steady state $(\bar{p}, \mathrm{Tanh}[\beta C/2])$ is globally stable, while for $\beta > \beta_1$ the steady state is an unstable saddle point with eigenvalues 0 and*

$$\lambda(\beta) = -\frac{b(1 - m^*(\beta))}{2B + b(1 + m^*(\beta))}.$$

   *At the critical value $\beta_1$ the steady state value $m^*(\beta_1) = -B/b$.*

3. *When the steady state is unstable, there exists a locally unique period 2 orbit $\{(\tilde{p}, \tilde{m}), (-\tilde{p}, \tilde{m})\}$ with $\tilde{m} = -B/b$. There exists a $\beta_2 > \beta_1$ such that the period 2 cycle is stable for $\beta_1 < \beta < \beta_2$.*

The proof of Proposition 3, therefore, is to show the condition under which these dynamic outcomes satisfy the condition specified in Proposition 2 for all periods.

**1:** When $C = 0$, the rational predictor is chosen by all agents whenever $p_{-1} \neq \bar{p}$. Therefore, this case is identical to the homogeneous expectations case. The relevant condition is $b/B < 1$, which is not satisfied by assumption. If $p_{-1} = \bar{p}$, then all agents select the naive predictor.

**2:** When $C > 0$, the steady state of the economy is $(\bar{p}, Tanh[-\beta C/2])$, where $\bar{p} = A/(B + b)$. By Proposition 2, the condition for strong rationality in this case is

$$\frac{b}{B}\left(\frac{1 + Tanh[-\beta C/2]}{2}\right) < 1.$$

Rearranging reveals that the condition is always satisfied provided $Tanh[-\beta C/2] < 2B/b - 1$.

**3:** When $C > 0$ and $\beta_1 < \beta < \beta_2$, ARED is period 2 cycle given by $\{(\tilde{p}, \tilde{m}), (-\tilde{p}, \tilde{m})\}$ with $\tilde{m} = -B/b$. Following Proposition 2, the condition for strong rationality is

$$\frac{b}{B}\left(\frac{1-\frac{B}{b}}{2}\right) < 1.$$

Rearranging reveals that this condition is satisfied provided $b/B < 3$. Therefore, for $\beta_1 < \beta < \beta_2$ and $1 < b/B < 3$, the period 2 cycle is a strongly rational outcome. $\square$

Simulations reveal that initial beliefs do not need to start at steady state or on the period 2 cycle in order for strong rationality to hold. There are many initial conditions for which the proportion of rational agents is sufficiently low before convergence to the attractor that strong rationality holds in each time period.